Monodromies for the Rabi Model

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Rabi Model - 1936

A model with one boson mode, $[a, a^\dagger] = 1$, coupled with a fermion mode, $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$,

$$H_R = a^\dagger a + \Delta\sigma_3 + g\sigma_1(a^\dagger + a) = \begin{pmatrix} a^\dagger a + \Delta & g(a^\dagger + a) \\ g(a^\dagger + a) & a^\dagger a - \Delta \end{pmatrix}.$$

Δ is associated with the level separation of the fermion mode.

g is the boson-fermion coupling.

The spectrum of H_R was found by D. Braak in 2011.

Jaynes-Cummings

The Jaynes-Cunnings model is an approximation of Rabi model. Indeed, consider

$$H_{JC} = a^\dagger a + \Delta\sigma_3 + g\left(\sigma^+ a + \sigma^- a^\dagger\right), \qquad (1)$$

and

$$H_{\overline{JC}} = a^\dagger a + \Delta\sigma_3 + g\left(\sigma^- a + \sigma^+ a^\dagger\right). \qquad (2)$$

Then

$$H_R = \frac{1}{2}\left(H_{JC} + H_{\overline{JC}}\right) = a^\dagger a + \Delta\sigma_3 + g\sigma_1(a^\dagger + a). \qquad (3)$$

Notice that $\sigma_1 = (\sigma^+ + \sigma^-)/2$.

Furthermore, $[H_{JC}, H_{\overline{JC}}] \neq 0$.

The Bargmann Realization

Let $\mathcal{H}$ be the Bargmann-Hilbert space of entire functions with scalar product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(z)}\, g(z)\, e^{-|z|^2}\, dxdy, \qquad z = x + iy. \tag{4}$$

On $\mathcal{H}$,

$$a^\dagger \;\mapsto\; z, \qquad a \;\mapsto\; \partial_z \equiv \frac{d}{dz},$$

so that $[a, a^\dagger] f(z) = f(z)$.

The Bargmann Realization

The problem is to solve

$$H_R\,\psi = E\,\psi, \qquad \psi = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{5}$$

In the Bargmann realization,

$$\begin{aligned} \partial_w f_+ &= \frac{E - gw}{w + g} f_+ - \frac{\Delta}{w + g} f_- \\ \partial_w f_- &= -\frac{\Delta}{w - g} f_+ + \frac{E + gw}{w - g} f_-, \end{aligned} \tag{6}$$

with $f_\pm = f_1 \pm f_2$.

The Bargmann Realization

The previous system is equivalent to a second order linear differential equation for, say f_+. Indeed, by setting

$$w \;\mapsto\; z = \frac{g-w}{2g}, \qquad f_+(z) = \; e^{-gw}\;\chi(z),$$

one obtains a confluent Heun equation

$$\partial_z^2\chi + \left(A + \frac{B}{z} + \frac{C}{z-1}\right)\partial_z\chi + \left(\frac{D}{z-1} + \frac{F}{z(z-1)}\right)\chi = 0, \quad (7)$$

with the following definitions, $\mathcal{E} = E + g^2$,

$$A = 4g^2, \qquad B = -\mathcal{E}, \qquad C = 1 - \mathcal{E},$$

$$D = -4g^2\mathcal{E}, \qquad F = \mathcal{E}^2 - \Delta^2$$

The Spectrum

At each regular singular point $z_i = 0, 1$, $i = 0, 1$, there are two independent local solutions.

We consider local solutions of the form

$$\chi_i(z) = (z - z_i)^{\alpha_i}\ \phi_i(z), \qquad \phi_i \text{ is analytic near } z_i.$$

Now, for the particular case of our confluent Heun equation,

$$\chi_0(z) = \mathrm{HeunC}(a_0; z), \qquad \chi_1 = \mathrm{HeunC}(a_1; 1 - z), \tag{8}$$

with $a_0 = (A, B, C, D, F)$ and $a_1 = (-A, C, B, -D, D + F)$.

The Spectrum

The Wronskian writes

$$W(\Delta, g, \mathcal{E};\ z) = \det \begin{pmatrix} \chi_1(z) & \chi_2(z) \\ \partial_z \chi_1(z) & \partial_z \chi_2(z) \end{pmatrix}. \tag{9}$$

Fixing g, Δ, and $\mathcal{E} \equiv E + g^2 \notin \mathbb{N}$, the spectrum is obtained by looking for the zeros of $W(\Delta, g, \mathcal{E};\ z)$.

This defines a complicated transcendental equation solvable by graphical methods.

For the case $\mathcal{E} = n \in \mathbb{N}$, corresponding to the Judd eigenstates B. Judd, 1979, the confluent Heun function becomes a polynomial of degree $(n - 1)$.

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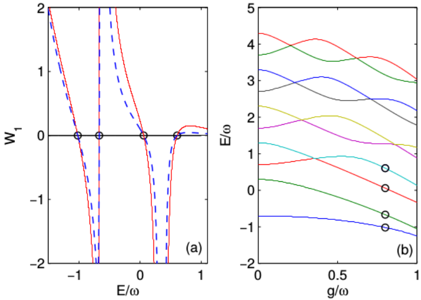
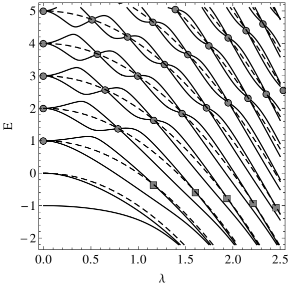
The Spectrum



(a) Batchelor et al. 2013, $z = 0(s), 0.5(d),\ \Delta = 0.7, g = 0.8$



(b) Macijewisky et al. 2013, $\Delta = 1, \lambda = g$

Integrability vs Solvability: the Problem

D. Braak, 2011 obtained the previous spectrum while coining a a new criterion of quantum integrability. This led to some controversy.

M. Batchelor and H-Q. Zhou, 2014 attempted to check the integrability of Rabi model in the Yang-Baxter sense. They could find monodromies satisfying Yang-Baxter relations only in two points of the parameter space.

Quantum integrability is quite complicated subject. For a review, see J. Caux and J. Mossel, Remarks on the notion of quantum integrability, 2011 . A nice work, see J. Clemente-Gallardo and G. Marmo, Towards a definition of quantum integrability, 2009

We showed in our work arXiv:1508.01342 that the difficulty in finding the monodromies is attached to the emergence of the Stokes' phenomenon.

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Standard Garnier Form

Set the fundamental matrix

$$\Phi(z) = \begin{pmatrix} f_+^{(1)} & f_+^{(2)} \\ f_-^{(1)} & f_-^{(2)} \end{pmatrix}. \tag{10}$$

The Rabi eigenvalue problem, $H_R\,\psi = E\psi$, can be cast as

$$\frac{d\Phi}{dz} = \left(\frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} \right) \Phi, \tag{11}$$

with $t = -4g^2$ and

$$A_0 = \begin{pmatrix} E+g^2 & -\Delta \\ 0 & 0 \end{pmatrix}, \qquad A_t = \begin{pmatrix} 0 & 0 \\ -\Delta & E+g^2 \end{pmatrix}. \tag{12}$$

Regular Singular Points

The system above has two regular singular points at $z = 0$ and $z = t$.

Let us consider the solution near $z = t$. The monodromy matrix M_t is obtained by

$$\Phi((z-t)e^{2\pi i} + t) = \Phi(z)M_t. \tag{13}$$

Indeed, near a regular singular point

$$\Phi(z)\big|_{z\approx t} = \begin{pmatrix} (z-t)^{\alpha_t^+} & 0 \\ 0 & (z-t)^{\alpha_t^-} \end{pmatrix}, \tag{14}$$

where $\alpha_t^\pm$ are the solutions of the indicial equation (Frobenius method).

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Regular Singular Points

A general monodromy near $z = t$ may then be written as

$$M_t = C_t \begin{pmatrix} e^{2\pi i \alpha_t^+} & 0 \\ 0 & e^{2\pi i \alpha_t^-} \end{pmatrix} C_t^{-1}, \tag{15}$$

where $C_t \in \mathrm{SL}(2, \mathbb{C})$ is called the connection matrix at t.

Analogous consideration leads us to M_0 and C_0.

Irregular Singular Points

The point $z = \infty$ is a irregular singular point.

The solution near this point depends on the direction. So, we set sectors on $\mathbb{C}$ by

$$\mathcal{S}_j = \{z \in \mathbb{C} \,|\, (2j-5)\frac{\pi}{2} < \arg z < (2j-1)\frac{\pi}{2}\}, \qquad j = 1, 2, 3, \ldots.$$

On each sector, the (asymptotic) solution near $z = \infty$ behaves as

$$\Phi_j(z) \equiv \Phi(z)\big|_j = G_j(z^{-1})\; e^{\frac{1}{2}z\sigma_3}\; z^{-\frac{1}{2}\theta_\infty\sigma_3}, \tag{16}$$

with $G_j(z^{-1}) = \mathbb{1} + \mathcal{O}(z^{-1})$.

This is the celebrated Stokes phenomenon.

The Stokes Parameter

The Stokes matrices relates asymptotic solutions between different sectors,

$$\Phi_{j+1}(z) = \Phi_j(z) S_j. \tag{17}$$

Now,

$$\Phi_j(e^{2\pi i z}) = \Phi_{j+2}(z)\ e^{-i\pi\theta_\infty \sigma_3}, \tag{18}$$

that is, S_{j+2} is identified with 2π rotations. Then, it is enough to choose the basis

$$S_{2j} = \begin{pmatrix} 1 & s_{2j} \\ 0 & 1 \end{pmatrix}, \qquad S_{2j+1} = \begin{pmatrix} 1 & 0 \\ s_{2j+1} & 1 \end{pmatrix}, \tag{19}$$

where s_k are known as Stokes parameters.

The Monodromy Data

At sector $\mathcal{S}_j$, the monodromy at infinity is defined by

$$M_\infty|_{\mathcal{S}_j} = S_j S_{j+1} e^{i\pi\theta_\infty\sigma_3}. \tag{20}$$

The monodromies M_0, M_t, M_∞ define a group with the usual matrix multiplication satisfying the relation

$$M_\infty M_t M_0 = \mathbb{1}. \tag{21}$$

The full monodromy data is $\{\theta_0, \theta_t, \theta_\infty, s_1, s_2\}$.

The Isomonodromy Method

The problem is now to find the full monodromy data for the Rabi model. We already know $\theta_0 = \theta_t = E + g^2$ and $\theta_\infty = 0$.

It remains to find the Stokes parameter $\vec{\sigma} = (s_1, s_2)$.

For that we consider the isomonodromy method.

We extend the Garnier system,

$$\frac{d}{dz}\Phi = \mathcal{A}_z(z,t)\Phi, \qquad \mathcal{A}_z(z,t) = \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t}, \tag{22}$$

so that the monodromy data is preserved. We also define

$$\mathcal{A}_t(z,t) = -\frac{A_t(t)}{z-t}. \tag{23}$$

We have a $SL(2,\mathbb{C})$ gauge potential $\mathcal{A} = (\mathcal{A}_z, \mathcal{A}_t)$.

The Isomonodromy Method

Integrability, that is, the preservation of the monodromy data is equivalent to the vanishing of the gauge curvature

$$F \equiv d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \partial_t \mathcal{A}_z - \partial_z \mathcal{A}_t + [\mathcal{A}_z, \mathcal{A}_t] = 0. \qquad (24)$$

This means that the $A_i(t)$'s, $i = 0, t$, satisfy

$$\frac{\partial A_0}{\partial t} = \frac{1}{t}[A_t, A_0], \qquad (25)$$

$$\frac{\partial A_t}{\partial t} = -\frac{1}{t}[A_t, A_0] - \frac{1}{2}[A_t, \sigma_3]. \qquad (26)$$

This set, dubbed Schlesinger system, defines a Lax pair for the isomonodromy flow.

It is equivalent to the existence of a special nonlinear differential equation known as Painleve 5.

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Painleve Equations

A singularity of the solution of an ODE is dubbed movable if it depends on the initial conditions. This means that such singularities are not predicted by the coefficients of the ODE.

An ODE has the Painleve property if the only movable singularities of the solutions are poles.

There are 6 non-trivial Painleve equations, P1,...,P6, that are not reducible to known equations. Painleve, 1898; Gambier, 1909

Their solutions are known as Painleve transcendents.

Heun and Painleve

The structure unveiled here is general for any of the Heun equations and the Painleve equations.

If one writes the Heun equations as

$$\mathfrak{H}(z,p_z;t)\psi = \frac{1}{f(t)}\left(P_0(z,t)\;p^2 + P_1(z,t)\;p + P_2(z,t)\right)\psi = \lambda\;\psi,$$

then the classical equation of motion associated with $\mathfrak{H}$, modulo some rescalings, is a Painleve equation.

For the confluent Heun, the Hamiltonian reads

$$\mathfrak{H} = -\frac{1}{t}\Big[z(z-1)\,p_z^2 + \big(-tz(z-1)+B(z-1)+Cz\big)p_z + Dtz\Big]. \quad (27)$$

Jimbo, Miwa, 1980

Finding the Stokes' Parameters

Jimbo, Miwa, 1980 wrote the Painleve 5 equation for the "τ function"

$$\frac{d}{dt}\log\tau(t,\vec{\theta},\vec{\sigma}) = -\frac{1}{2}\mathrm{Tr}\sigma_3 A_t - \frac{1}{t}\mathrm{Tr}A_0 A_t. \tag{28}$$

The Stokes' parameters $\vec{\sigma} = (s_1, s_2)$ are found implicitly by the system defined by the initial conditions for the "τ function",

$$\frac{d}{dt}\log\tau(t,\vec{\theta},\vec{\sigma})\Big|_{t=-4g^2} = \frac{E+g^2}{2} + \frac{\Delta^2}{4g^2},$$

$$\frac{d^2}{dt^2}\log\tau(t,\vec{\theta},\vec{\sigma})\Big|_{t=-4g^2} = \frac{1}{t^2}\mathrm{Tr}A_0 A_t = \frac{\Delta^2}{16g^4}.$$

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Lessons

The class of Heun functions, generalizing the ${}_2F_1$ functions, is a powerful tool.

A proper assessment of the singular points of ODE's may point out to new integrability classes.

We can also obtain the spectrum of the Rabi model using the Painleve 5. It is amenable to symbolic/numeric computations.

The method shown here is quite general and can be applied to more general Rabi models. For instance, the $\mathbb{Z}_2$ symmetry breaking model proposed by Braak

$$H_B = H_R + \varepsilon\, \sigma_1. \tag{29}$$

Other uses of Heun Equations: Scattering in Kerr Background

The Kerr black hole is described by mass M and spin $a = J/M$. It contains two horizons

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \tag{30}$$

The Klein-Gordon equation for a massless scalar field on this background is

$$\Box\Phi \equiv \frac{1}{\sqrt{-g}}\partial_{\mu}\sqrt{-g}g^{\mu\nu}\partial_{\nu}\Phi = 0. \tag{31}$$

After separation of variables, $\Phi(t, r, \theta, \varphi) = e^{-i\omega t + m\varphi}R(r)S(\theta)$, the equations for the spheroidal function $S(\theta)$ and the radial function $R(r)$ are of the confluent Heun type.

Anharmonic Oscillator and Triconfluent Heun

The anharmonic oscillator equation reads

$$\Big(-\partial_z^2 + \frac{1}{4}(z^2+b)^2 + (a-1)z \Big)\psi(z) = E\psi(z) \tag{32}$$

This equation may be obtained from the triconfluent Heun equation, with one irregular singular point,

$$\Big(-\partial_z^2 + (z^2+b)\partial_z + az \Big)\Phi(z) = E\Phi(z), \tag{33}$$

via the transformation

$$\Phi(z) = e^{\frac{z^3}{6}+\frac{b}{2}z}\ \psi(z). \tag{34}$$

¡MUCHAS GRACIAS!
MUITO OBRIGADO!