Filiberto Ares
Departamento de Física Teórica
Universidad de Zaragoza

Martes Cuántico

Entanglement, entropy and fermionic chains

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Outline

1. What is entanglement?
2. How can we quantify bipartite entanglement?
3. Our system: fermionic chains

What is entanglement? The singlet state

- Bipartite quantum system with $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, e. g. two qubits

  Each part in two possible states: $|0\rangle_i\,, \quad |1\rangle_i\,, \quad i = X, Y$

- Singlet state:

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_X \otimes |1\rangle_Y - |1\rangle_X \otimes |0\rangle_Y\right)$$

- Measurement of, e.g., X: $\begin{cases} 50\% & |1\rangle_X \otimes |0\rangle_Y \\ 50\% & |0\rangle_X \otimes |1\rangle_Y \end{cases}$

- The singlet state is an example of entangled state.
  - The outcome in one part determines the result of a measurement in the other part
  - This two measurements can be separated by a space-like interval (EPR paper)
  - There isn't a causal connection between them

What is entanglement?

- This is "*not one but rather the characteristic trait of Quantum Mechanics*" (Schrödinger, 1935)

- There isn't a classical analogue of this quantum correlations

- Entanglement reflects "*the best possible knowledge of a whole doesn't necessarily include the best possible knowledge of its parts*" (Schrödinger, 1935)

- In the later system:
  - Singlet state: $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_X \otimes |1\rangle_Y - |1\rangle_X \otimes |0\rangle_Y\right)$, entangled
  - Possible states after a measurement: $|0\rangle_X \otimes |1\rangle_Y \,, \quad |1\rangle_X \otimes |0\rangle_Y \,,$ separable states, not entangled

What is entanglement? Bipartite entanglement

- System composed by two subsystems X, Y: $\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y$

- Schmidt decomposition: $|\Psi\rangle \in \mathcal{H}$ can be written

$$|\Psi\rangle = \sum_{l=1}^{\nu} \sqrt{p_l}\, |X_l\rangle \otimes |Y_l\rangle\,, \quad \nu \le d = \min(\dim(\mathcal{H}_X), \dim(\mathcal{H}_Y))$$

$\{|X_l\rangle\}, \{|Y_l\rangle\}$: orthonormal sets in $\mathcal{H}_X, \mathcal{H}_Y$, normalization: $\sum_{l=1}^{\nu} p_l = 1$

- ν is the Schmidt rank
  - If $\nu = 1 \Rightarrow |\Psi\rangle = |X\rangle \otimes |Y\rangle$ (separable)
  - If $\nu > 1 \Rightarrow |\Psi\rangle \neq |X\rangle \otimes |Y\rangle$ (entangled)

- ν seems a good measure of entanglement however... e.g.
  - Singlet state: $p_1 = p_2 = 0.5$, $\nu = 2$
  - Other state with $p_1 = 0.99$, $p_2 = 0.01$, $\nu = 2$

but singlet state appears more entangled than the another one!

How can we quantify the bipartite entanglement?

- We need a better magnitude to quantify the entanglement
- Bipartite systems in a pure state $|\Psi\rangle$: Rényi entanglement entropy

$$S_\alpha = \frac{1}{1-\alpha} \log \Omega_\alpha; \quad \Omega_\alpha = \sum_{l=1}^{\nu} p_l^\alpha; \quad \alpha > 1$$

In the limit $\alpha \to 1$, von Neumann entanglement entropy:

$$S_1 = -\sum_{l=1}^{\nu} p_l \log p_l$$

  - Non ent.: $\nu = 1 \Rightarrow \Omega_\alpha = 1 \Rightarrow S_\alpha^{\text{min}} = 0$
  - Max. ent.: $\nu = d$ and $p_l = \frac{1}{d} \Rightarrow \Omega_\alpha = d^{1-\alpha} \Rightarrow S_\alpha^{\text{max}} = \log d$

- In the later examples
  - Singlet: $\Omega_\alpha = 2^{1-\alpha} \Rightarrow S_\alpha = \log 2$, max. entangled
  - $p_1 = 1 - \varepsilon$, $p_2 = \varepsilon$: if $\varepsilon \to 0$, $\Omega_\alpha \approx 1 - \alpha\varepsilon \Rightarrow S_\alpha \approx \frac{\alpha}{\alpha - 1}\varepsilon$

How can we quantify the bipartite entanglement?

- S_α can be defined without Schmidt decomposition

- Density matrix formalism: total system state

$$\rho = |\Psi\rangle\langle\Psi| = \sum_{i,j} \sqrt{p_i p_j}\, |X_i\rangle\langle X_j| \otimes |Y_i\rangle\langle Y_j|$$

- State of X

$$\rho_X = \mathsf{Tr}_Y(\rho) = \sum_l p_l\, |X_l\rangle\langle X_l|$$

- Therefore

$$\Omega_\alpha = \mathsf{Tr}(\rho_X^\alpha) = \sum_l p_l^\alpha$$

- Rényi entanglement entropy

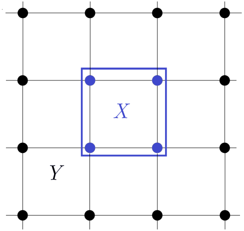
$$S_\alpha = \frac{1}{1-\alpha}\,\mathsf{Tr}(\rho_X^\alpha)$$

In the limit $\alpha \to 1$, von Neumann entanglement entropy

$$S_1 = -\,\mathsf{Tr}(\rho_X \log \rho_X)$$

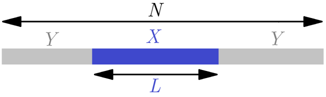
How can we quantify the bipartite entanglement?

- Area law: ground state entanglement $\propto$ bonds broken isolating subsystem. E.g. in a 2-D lattice:



It's violated in excited states!

- Unidimensional systems: from (1+1) conformal field theory, in the ground state of a local, gapless, Hamiltonian



$$S_\alpha \approx \frac{1+\alpha}{\alpha}\frac{c}{6}\log L + C_\alpha; \quad N \to \infty;$$

(Holzhey, Larsen, Wilczek, hep-th/9403108)
(Calabrese & Cardy, hep-th/0405152)

c: central charge of the underlying CFT

Our system: fermionic chains

- Unidimensional chain of N fermions: $\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$
- Over each site: a_n, $a_n^\dagger$, $\{a_n, a_m^\dagger\} = \delta_{nm}$, $\{a_n, a_m\} = \{a_n^\dagger, a_m^\dagger\} = 0$

- Consider the Hamiltonian is like

$$H = \sum_{n=1}^{N} \sum_{r=1}^{N/2} J_r a_n^\dagger a_{n+r} + h.c.$$

- Its eigenstates $H \left|\Psi_{\mathcal{K}}\right\rangle = E_{\mathcal{K}} \left|\Psi_{\mathcal{K}}\right\rangle$ are

$$\left|\Psi_{\mathcal{K}}\right\rangle = \prod_{k\in\mathcal{K}} b_k^\dagger \left|0\right\rangle ; \quad \mathcal{K} : \text{set b-excitations}$$

b-operators: Fourier transform of a-operators

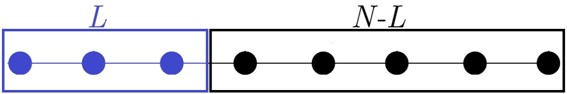
$$b_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{\frac{2\pi i k n}{N}} a_n; \quad k = -N/2, ..., N/2 - 1$$

- Energy of $\left|\Psi_{\mathcal{K}}\right\rangle$: $E_{\mathcal{K}} = \sum_{k\in\mathcal{K}} \Lambda_k$

$$\Lambda_k = \sum_{r=1}^{N/2} J_r e^{\frac{2\pi i k r}{N}} + c.c.$$

Our system: fermionic chains

- Take now $X = \{1, ..., L\}; Y = \{L+1, ..., N\} \Rightarrow S_\alpha = \text{Tr}(\rho_X^\alpha)/(1-\alpha)$?



- We must know the spectrum of $\rho_X = \text{Tr}_Y(|\Psi_\mathcal{K}\rangle\langle\Psi_\mathcal{K}|)$ with dimension $2^L \Rightarrow$ for large L, its diagonalization is complicated!

- ρ_X satisfies Wick theorem $\Rightarrow$ Peschel's method can be employed (Peschel, cond-mat/0212631)

  $\text{Tr}(\rho a_n^\dagger a_m^\dagger a_p a_q) = \text{Tr}(\rho a_n^\dagger a_q)\,\text{Tr}(\rho a_m^\dagger a_p) - \text{Tr}(\rho a_n^\dagger a_p)\,\text{Tr}(\rho a_m^\dagger a_q) + \text{Tr}(\rho a_n^\dagger a_m^\dagger)\,\text{Tr}(\rho a_p a_q)$

- Then S_α can be expressed in terms of $C_{nm} = \text{Tr}(\rho_X a_n^\dagger a_m)$

$$S_\alpha = \frac{1}{1-\alpha}\,\text{Tr}\log[(I-C)^\alpha + C^\alpha]$$

- We only have to diagonalize C_{nm}, a $L \times L$ matrix. Now we can use numeric methods and reach larger L

Our system: fermionic chains | State 1

- Consider the Tight Binding Model Hamiltonian

$$H = -T \sum_{n=1}^{N} a_n^\dagger (a_{n-1} + a_{n+1}); \quad T > 0$$

- Here we have

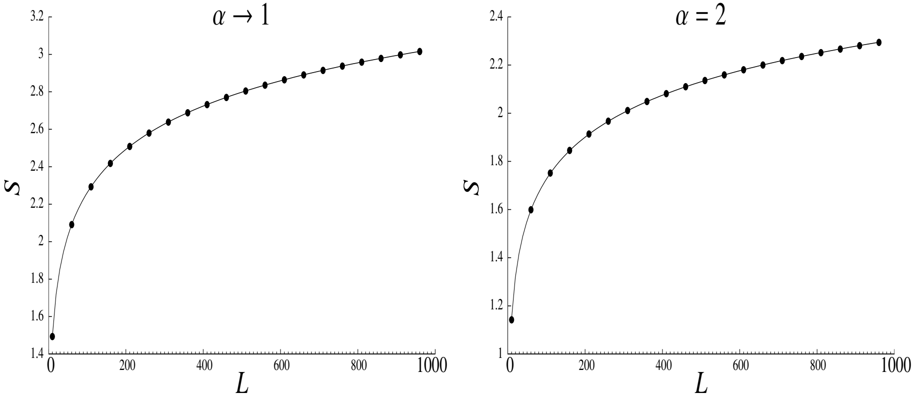
$$H \left|\Psi_{\mathcal{K}}\right\rangle = \sum_{k \in \mathcal{K}} \Lambda_k^{\mathsf{TBM}} \left|\Psi_{\mathcal{K}}\right\rangle; \quad \Lambda_k^{\mathsf{TBM}} = -2T \cos\left(\frac{2\pi k}{N}\right)$$

- Its ground state is the State 1: sites $|k| < N/4$: occupied, sites $|k| > N/4$: empty

$$\left|\Psi_{\mathcal{K}_1}\right\rangle = \prod_{\Lambda_k^{\mathsf{TBM}} < 0} b_k^\dagger \left|0\right\rangle; \quad \mathcal{K}_1 = \{0...01...111...10...0\}$$

Our system: fermionic chains | State 1 (II)

Numeric entropies for $N = 10^6$ varying the length of interval L



$S_1 \approx \frac{1}{3} \log L + 0.7266$ $\qquad$ $S_2 \approx \frac{1}{4} \log L + 0.570234$

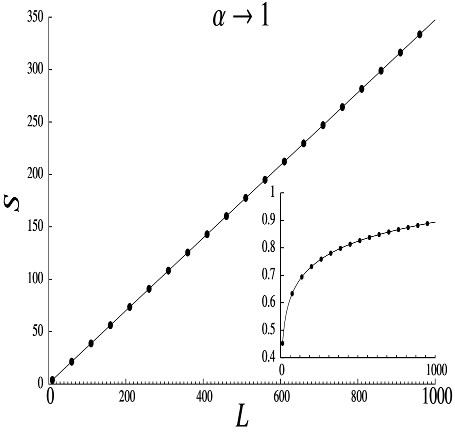
(From Ares, Esteve, Falceto, Sánchez-Burillo, quant-ph/1401.5922)

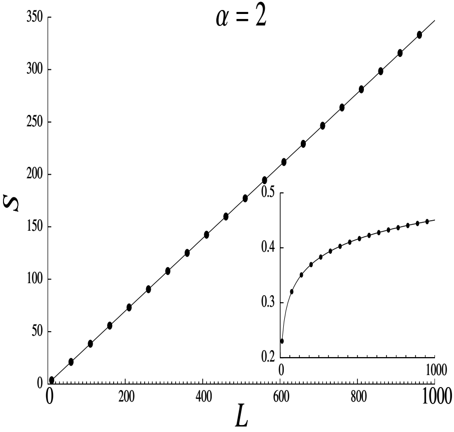
Our system: fermionic chains | State 2

State 2: sites $|k| < N/4$: occupied alternatively, sites $|k| > N/4$: empty

$$\mathcal{K}_2 = \{0...01010....01010...0\}$$

Numeric entropies for $N = 10^6$ varying the length of interval L





$S_1 \approx \frac{\log 2}{2} L + 0.10067 \log L + 0.220768 \qquad S_2 \approx \frac{\log 2}{2} L + 0.05033 \log L + 0.114270$

(From Ares, Esteve, Falceto, Sánchez-Burillo, quant-ph/1401.5922)

Our system: fermionic chains | Results

- Eigenstates quadratic, translational invariant Hamiltonian

$$|\Psi_{\mathcal{K}}\rangle = \prod_{k\in\mathcal{K}} b_k^\dagger |0\rangle$$

- S_α of an interval L shows different behaviors

  - $S_\alpha \approx B_\alpha \log L + C_\alpha$

    $|\Psi_{\mathcal{K}}\rangle$: ground state local, gapless Hamiltonian (CFT prediction)
    E.g. State 1, g.s. of TBM

  - If the Hamiltonian has a gap ($|GS\rangle = |0\rangle$), $S_\alpha = 0$

  - Interpretation of $|\Psi_{\mathcal{K}}\rangle$ where

$$S_\alpha \approx A_\alpha L + B_\alpha \log L + C_\alpha ?$$

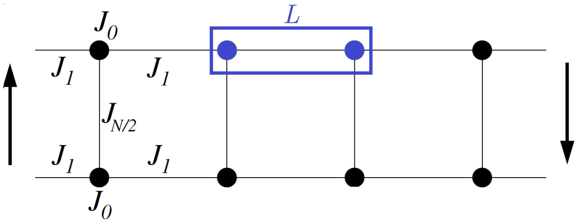
    E. g. State 2

Our system: fermionic chains | Ladders

- State 2 can be seen as the ground state of, e. g.

$$H = \sum_n J_0 a_n^\dagger a_n + J_1 a_n^\dagger a_{n+1} + J_{N/2} a_n^\dagger a_{n+N/2} + h.c.$$

$$J_{N/2} = J_0, \quad J_1 = -2J_0$$



- *Area law:* entanglement $\propto$ bonds broken isolating subsystem

Agreement with our results: $S_\alpha \propto L =$ bonds broken

Conclusions and remarks

- Pure state, bipartite entanglement can be measured by Rényi entropy of the reduced density matrix

- Rényi entropy of a block L in a fermionic chain described by a quadratic, translational invariant Hamiltonian.

- For its eigenstates:

$$\begin{cases} S_\alpha \propto \log L & \text{g.s. gapless local Hamiltonian} \\ S_\alpha \propto L & \text{g.s. non local Hamiltonian} \end{cases}$$

- These results can be obtained analytically (Ares, Esteve, Falceto, Sánchez-Burillo, quant-ph/1401.5922)

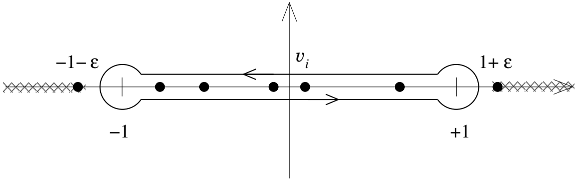
- Disjoint intervals?

Our system: fermionic chains | Analytics?

- Integral expression for S_α applying residue's theorem

$$S_\alpha = \frac{1}{2\pi i(1-\alpha)} \lim_{\varepsilon\to 0^+} \oint_{\mathcal{C}} \log\left[\left(\frac{1+\varepsilon+z}{2}\right)^\alpha + \left(\frac{1+\varepsilon-z}{2}\right)^\alpha\right] \frac{d\log D_V(z)}{dz} dz$$

$$D_V(z) = \det(zI - V), \quad V = 2C - I; \quad v_i : V \text{ eigenvalues}$$



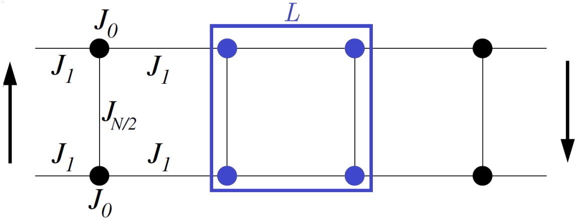
- For $|\Psi_{\mathcal{K}}\rangle$, C_{nm}, V_{nm} only depend on $n-m \Rightarrow$ Toeplitz matrices
- Fisher-Hartwig theorem for Toeplitz determinants: for $L \gg 1$ and $N \to \infty$

$$\log D_V(z) \approx A(z)L + B(z)\log L + C(z) + \cdots$$

Analytical computation of $A_\alpha, B_\alpha, C_\alpha$
(Firstly, ground state: Jin & Korepin, quant-ph/0304108)
(In general: Ares, Esteve, Falceto, Sánchez-Burillo, quant-ph/1401.5922)

Ladders (bis)

- And if we take a *fragment* of the ladder?



$\equiv$ two disjoint blocks in a chain separated by $N/2 - L$ sites

- Entanglement entropy of the fragment

$$S_\alpha = \frac{1+\alpha}{\alpha}\frac{1}{6}\log L + C'_\alpha$$