Entanglement of Identical Particles

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1 Identical Particles

2 Statement of the Problem

3 Entanglement for Identical Particles

4 Examples
My friend Prof. A. Polito (IF-UnB) works on the nature of space in Leibnitz. I learned from him about a key principle due to Leibnitz.

**Identity of indiscernibles**: two things are identical if and only if they have the very same (intrinsic) properties regardless of their distinct positions in space.

I think this principle seems to lead to:

- the conceptual distinction between internal (intrinsic) versus external (extrinsic) properties;
- the principle of symmetry; in particular, **gauge symmetry**.


http://www.sbf1.sbfisica.org.br/eventos/snef/xx/programa/resumo.asp?insId=175&traId=1
The Gibbs Paradox

Statement:

- The mixing of two identical gases does not produce entropy: \( \Delta S = 0 \).

- The difference between the entropy of two separated distinct gases and the entropy of the mixing of these gases is

\[
\Delta S = S_A + S_B - S_{AB} \\
= -N_A \log N_A - N_B \log N_B + (N_A + N_B) \log(N_A + N_B).
\]

- If \( A = B \), then \( \Delta S = 2N_A \ln 2 \).

Identical Particles

The Gibbs Paradox

Another statement:

- For ideal gases, $E = \frac{3}{2}NT + \text{cte.}$, $PV = NT$, then from $T\Delta S = \Delta E + P\Delta V$ one gets

$$S(P, T) = \frac{5}{2}N \log T - N \log P + C(N).$$

- The function $C(N)$ does not depend on $P, T$. But the way it is usually fixed as a full-constant leads to wrong results if not used consistently.

The resolution of the paradox: the counting of identical particle microstates is different from the counting of distinct particle microstates. A factor of $N!$ from the group of permutations $S_N$ has to be accounted for.

Identical Particles

Configuración Espacial

- Defina $\mathbb{R}^k$, $k \geq 3$.
- Dé $N \geq 2$ partículas idénticas sin spin.
- Dé $x_j \in \mathbb{R}^k$ para denotar la posición del $j$-ésimo átomo.
- Un sistema de $N$-átomos se define como la configuración no ordenada
  $$\tilde{q} = \{x_1, x_2, \ldots, x_N\} = \{x_2, x_1, \ldots, x_N\} \in \tilde{Q}$$

Si dos átomos no pueden ocupar la misma posición en el espacio, debemos quitar la diagonal
$$\Delta = \{x_i = x_j | i \neq j\}.$$  La configuración espacio es $Q = \tilde{Q}\setminus\Delta$. Téngase en cuenta la topología complicada.

The fundamental group of $Q$ is $\pi_1(Q) = S_N$, the group of permutation of $N$ objects. This leads to fermion and boson statistics.

A transposition in $s_{ij} \in S_N$ may be represented by a loop $\gamma_{ij}(t)$ in $\widetilde{Q}$ that permutes particle $x_i$ and $x_j$.

Note that if $\mathbb{R}^2$, then $\pi_1(Q) = B_N$, the group of braids. This leads to anyons or fractional statistics.
To quantize a system of identical particles, we would like to consider a Hilbert space on top of $Q$. This is hard to directly construct.

The best strategy is to consider a Hilbert space $\mathcal{H}$ on top of a simply-connected space $\overline{Q} \supseteq Q$.

Decompose $\mathcal{H}$ into irreducible representations of $\pi_1(Q) = S_N$

$$\mathcal{H} = \bigoplus_l \mathcal{H}^{(l)}.$$  

Note the strong analogy to working with gauge symmetry.
O. Greenberg distinguishes between Spin-Statistics and Spin-Locality.

- **Statistics**: a field operator $\hat{\phi}$ may be decomposed as
  \[
  \hat{\phi}(x) = \sum_{k} \varphi_{k}(x) \hat{a}_{k} \quad \text{or} \quad \hat{\phi}(x) = \sum_{k} \psi_{k}(x) \hat{b}_{k},
  \]
  i.e., in terms of either bosons $[\hat{a}_{k}, \hat{a}_{j}] = 0$ or fermions $\{\hat{b}_{k}, \hat{b}_{j}\} = 0$.

- **Locality**: for space-like distance $(x - y)^2 < 0$, a field may be
  - local iff $[\phi(x), \phi(y)] = 0$,
  - anti-local iff $\{\phi(x), \phi(y)\} = 0$.

O. Greenberg expands an example due to Res Jost. He takes a neutral scalar field $\phi$.

- If $\phi$ is expanded in terms of fermion operators, then the corresponding observables are non-local.
  - Observe that even anti-commutators of such fields are non-local $\{\phi(x), \phi(y)\} = \Delta^{(1)}(x - y)$.
  - The field $\phi$ still satisfies the CPT-theorem.

- If $\phi$ is anti-local, like in Lüders-Zumino proof of spin-statistics theorem, then $\phi \equiv 0$. 

Indistinguishability
In 1986, Bombelli, Koul, Lee and Sorkin proposed and solved the following problem modeled after a black hole:

Consider a (scalar) field on a space-like hypersurface $\Sigma$. Integrate out the fields on a region $R \subset \Sigma$. What is the entropy emerging out from this process?

**Solution:** $S \propto$ Volume of $\partial R$.

Incidentally, Srednicki solved the same problem later in 1993.

Entanglement Entropy (Formal)

Consider $\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_L$. From a state vector $|\psi\rangle \in \mathcal{H}$, we form the (pure) density matrix $\rho : \mathcal{H} \to \mathcal{H}$ as

$$\rho = |\psi\rangle \langle \psi|.$$ 

The reduced density matrix is defined by the partial trace

$$\rho_R = \text{Tr}_L \rho.$$ 

The entanglement (a.k.a. von Neumann) entropy is

$$S = -\text{Tr} \rho_R \log \rho_R.$$ 

If $S \neq 0$, $\rho_R$ is a mixed state, that is, $\rho_R$ cannot be constructed out of state vectors in $\mathcal{H}_R$. 

Indistinguishability
Recently, R. Sorkin found a formula for entanglement entropy associated with a space-time region $R$ of some manifold $M$.

From the eigenvalues of the operator $iL = \Delta^{-1} W$ on $R$, with

- $W(x, x')$: Wightman function;
- $i\Delta(x, x') = W(x, x') - \overline{W}(x', x) = 2 \text{Im}W(x, x')$,

He obtained:

$$S = \text{Tr} \ L \log |L| \equiv \sum_{\lambda} \lambda \log \lambda. \quad (2)$$

What is the use of entropy?

What is the use of entropy formula or counting formula
\[ S = -\text{Tr} \hat{\rho} \log \hat{\rho} \]

What is the use of entanglement?

What is the relation between entanglement and entropy?

Besides those, what about entanglement, entrelazamiento, enredamiento, intrication or emaranhamento?
In physics of black holes, we have

- **Bekenstein-Hawking entropy** associated with a black hole
  \[ S = \frac{A_h}{4}; \]

- in some cases, we have a **counting formula**, BUT we do not know what are being counted.

**Problems:**

- What should be counted? In which conditions?

- Why should we care to count something into an entropy formula?

- What does we learn from such counting?
Comments on the Literature

General Comments


- Dyakonov, M. Quantum computing: a view from the enemy camp, Optics and Spectroscopy 95 261-267 (2003), cond-mat/0110326. It will be impossible to construct a quantum computer. It requires the control of $10^5$ particles.
1 Identical Particles

2 Statement of the Problem

3 Entanglement for Identical Particles

4 Examples
Statement of the Problem

Fermions in a Double-Well

Imagine a double-well or two quantum-dots. At each well $L$ or $R$ one finds electrons (qubits) with either spin $\uparrow$ or spin $\downarrow$. The 1-particle Hilbert space is spanned by

$$\{ |L, +\rangle, |L, -\rangle, |R, +\rangle, |R, -\rangle \}.$$

- Suppose initially one electron at each well.
- Problem: trace out d.o.f. associated to $R$-well. Equivalently, allow only observables associated to $L$-well.
- What is the entropy emerging out of this process?

Identical Fermions

Wave-functions do not overlap.

Wave-functions overlap. Use Slater determinant.

Identical Fermion

In a $N$-tuple-well system, with $N > 2$, filled with fermions, a new question arises.

Can we trace out some wells respecting (fermion) statistics?


Main problems:

- **Non-natural values for entropy**: e.g. $S \neq 0$ for separable cases.
- **Non-universal criteria**: different criteria for different statistics.
- **Focus on bosons and fermions**: e.g. no anyons.
Imagine a two-photons system in a Bell-like state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |h\rangle \otimes |v\rangle + |v\rangle \otimes |h\rangle \right),$$

with $|h\rangle$ and $|v\rangle$ standing for horizontal and vertical polarization.

- If the photons are distinguishable (e.g. different momenta though same frequency), then the Bell-like state is entangled.
- If the photons are indistinguishable, the Bell-like state seems to be separable.

1 Identical Particles

2 Statement of the Problem

3 Entanglement for Identical Particles

4 Examples
Partial trace fails to respect in a natural way correlations due to the indistinguishability (or identity) of particles.

There is an equivalent operation generalizing the notion of partial trace. Moreover it allows the treatment of entanglement of identical or non-identical particles on an equal footing:

Restriction of a state to a subalgebra.

Instead of density matrix $\rho : \mathcal{H} \to \mathcal{H}$, we regard a state as a linear functional on the algebra of observables $\mathcal{A}$. Indeed, from the expectation value $\langle O \rangle$ of the observable $O$

$$\omega_\rho (O) \equiv \langle O \rangle = \text{Tr} \ \rho O,$$

we abstract the notion of

a state on an algebra of observable $\mathcal{A}$

as a linear functional

$$\omega : \mathcal{A} \to \mathbb{C},$$

such that $\omega (1) = 1$ and $\omega (O^*O) \geq 0$, for any $O \in \mathcal{A}$. 
Entanglement for Identical Particles

Restriction to Subalgebra

The initial data to describe a quantum system is therefore

\((\mathcal{A}, \omega)\).

Consider a subalgebra \(\mathcal{A}_0 \subset \mathcal{A}\). Instead of partial trace, consider

\[\omega_0 \equiv \omega|_{\mathcal{A}_0},\]  \hspace{1cm} (5)

that is, the restriction of state \(\omega\) on \(\mathcal{A}\) to the subalgebra \(\mathcal{A}_0\).

Therefore,

entanglement of a subalgebra \(\mathcal{A}_0 \subset \mathcal{A}\) with the algebra \(\mathcal{A}\) for a state \(\omega\).

Gelfan’d-Naimark-Segal (GNS) construction gives a Hilbert space $\mathcal{H}_\omega$ out of $(\mathcal{A}, \omega)$ where $\mathcal{A}$ is represented on.

1. **Inner product** in $\mathcal{A}$ out of $\omega$: $\langle \alpha | \beta \rangle \equiv \omega(\alpha^* \beta)$.

2. It may exist **null states**: $\mathcal{N} = \{0 \neq n \in \mathcal{A} \mid \langle n | n \rangle = 0\}$.

3. **Removal of null states**: $\mathcal{H}_\omega = \mathcal{A}/\mathcal{N}$, that is, set of classes of equivalence

   \[ \tilde{\alpha} = \alpha + \mathcal{N}. \]

4. **Action of observable** $\alpha \in \mathcal{A}$ on $\mathcal{H}_\omega$: $\alpha |\tilde{\beta}\rangle = |\tilde{\alpha} \tilde{\beta}\rangle$. 
The vector $|\tilde{1}\rangle$, where $\tilde{1} = 1 + \mathcal{N}$, is dubbed cyclic vector.

A dense subset of $\mathcal{H}_\omega$ may be generated by the action of all $\alpha \in \mathcal{A}$.

Also, from the density matrix $\rho_\omega = |\tilde{1}\rangle \langle \tilde{1}|$, we obtain

$$\omega(\alpha) = \text{Tr} (\rho_\omega \alpha).$$

The Hilbert space $\mathcal{H}_\omega$ may be reducible w.r.t. $\mathcal{A}$, so that $\mathcal{H}_\omega = \bigoplus_i \mathcal{H}_i$. Thus, there exist projectors $P_i$, such that

$$|\tilde{1}\rangle = \sum_i P_i |\tilde{1}\rangle = \sum_i |\tilde{P}_i\rangle.$$. 
Density Matrix

A density matrix associated to state $\omega$ writes

$$\rho_{\omega} = \sum_{i} |\tilde{P}^i\rangle\langle\tilde{P}^i| = \sum_{i} \rho^i,$$

(9)

with corresponding entropy

$$S(\rho_{\omega}) = -\text{Tr} \: \rho_{\omega} \log \rho_{\omega}.$$  (10)

Equivalently, set normalized rank-1 density matrices

$$\hat{\rho}^i = \frac{1}{\lambda_i} \rho^i, \quad \lambda^i = \omega(P^i),$$

(11)

so that

$$S(\rho_{\omega}) = -\sum_{i} \lambda_i \log \lambda_i.$$  (12)
$M_2(\mathbb{C})$: 2 × 2 Matrices

A general element $a \in M_2(\mathbb{C})$ expands as

$$a = \sum_{i,j=1,2} a_{ij} |i\rangle \langle j| \equiv \sum_{ij} a_{ij} e_{ij} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (13)$$

Set a state on the algebra $M_2(\mathbb{C})$ as

$$\omega_{\lambda}(a) = \lambda a_{11} + (1 - \lambda) a_{22}, \quad 0 \leq \lambda \leq 1. \quad (14)$$

Observe,

$$\omega_{\lambda}(1) = 1 \quad (15)$$

$$\omega_{\lambda}(a^\dagger a) = \sum_k (\lambda |a_{k1}|^2 + (1 - \lambda) |a_{k2}|^2) \geq 0. \quad (16)$$
\textbf{GNS Construction}

\textbf{UnB} \quad M_2(C). \textbf{The } \lambda = 0 \textbf{ case.}

1. **Inner product** in $\mathcal{A}$ out of $\omega$: $\langle a | b \rangle = \omega_0 (a^\dagger b) = \sum_k \bar{a}_k b_{k2}$, $a, b \in \mathcal{A} \equiv \mathbb{C}^4$.

2. **Null states**: solutions $0 \neq a \in \mathcal{A}$ of $\omega_0 (a^\dagger a) = 0$ are spanned by $a_{k1}$. Thus, $\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} \in N_{\omega_0} \equiv \mathbb{C}^2$.

3. The **GNS Hilbert space** $\mathcal{H}_{\omega_0} = \mathcal{A} / N_{\omega_0} \equiv \mathbb{C}^2$ spanned by vectors

$$|\tilde{e}_{k2}\rangle = |e_{k2} + N_{\omega_0}\rangle.$$

4. **Action of** $\mathcal{A}$ on $\mathcal{H}_{\omega_0}$: $a|\tilde{b}\rangle = |\tilde{a}b\rangle$.

5. $|1\rangle = |\tilde{e}_{22}\rangle$. The GNS Hilbert space is **irreducible** w.r.t. $\mathcal{A}$. The density matrix is $\rho_{\omega_0} = |\tilde{e}_{22}\rangle \langle \tilde{e}_{22}|$. It has **entropy zero**.
\( M_2(\mathbb{C}). \text{ The } \lambda \neq 0, 1 \text{ case.} \)

1. **Inner product** on \( \mathcal{A} \) out of \( \omega \): for \( a, b \in \mathcal{A} \),

\[
\langle a|b \rangle = \omega_\lambda(a^\dagger b) = \sum_k (\lambda \overline{a}_k \overline{b}_k + (1 - \lambda) \overline{a}_k \overline{b}_k)
\]

2. There is **no non-trivial null states**.

3. The **GNS Hilbert space** is \( \mathcal{H}_{\omega_\lambda} = \mathcal{A} \equiv \mathbb{C}^4 \).

4. The action of \( a \in \mathcal{A} \) on \( |b\rangle \in \mathcal{H}_{\omega_\lambda} \) is \( a|b\rangle = |ab\rangle \). Now, there exist subspaces \( \mathcal{H}_i \equiv \mathbb{C}^2 \), \( i = 1, 2 \), such that

\[
a \cdot \mathcal{H}_i \subseteq \mathcal{H}_i \quad \forall a \in \mathcal{A}, \quad \mathcal{H}_{\omega_\lambda} = \mathcal{H}_1 \oplus \mathcal{H}_2.
\]
Entanglement for Identical Particles

Entanglement Entropy

The Hilbert space $\mathcal{H}_\omega$ may be reducible w.r.t. representations of $A_0$:

$$\mathcal{H}_\omega = \bigoplus_i \mathcal{H}_i.$$ (17)

Set $P_i : \mathcal{H}_\omega \rightarrow \mathcal{H}_i$ as orthogonal projectors. Then

$$|\tilde{1}_A\rangle = \sum_i P_i |\tilde{1}_A\rangle,$$ (18)

$$\mu_i = ||P_i |\tilde{1}_A\rangle||.$$ (19)

The entanglement entropy is

$$S(\omega, A_0) = - \sum_i \mu_i^2 \log \mu_i^2.$$ (20)
Identical Particles

- **One-particle Hilbert space**: \( \mathcal{H}^{(1)} = \mathbb{C}^d \). The group \( U(d) \) acts naturally on \( \mathbb{C}^d \).

- **Algebra of observables**: \( M_d(\mathbb{C}) \cong \mathbb{C}U(d) \).

- **\( k \)-particle Hilbert space**: \( \mathcal{H}^{(k)} = \bigotimes_{A,S}^k \mathcal{H}^{(1)} \).

- For \( g \in U(d) \), the **Coproduct** \( \Delta(g) = g \otimes g \) allows one to represent one-particle observables in the \( k \)-particle sector. Furthermore, it takes care of statistics.
  - Cf. Addition of angular momentum.

There are two main types of subalgebras we may be interested in a many-particles system:

1. The subalgebra of one-particle observables.

2. A subalgebra of partial one-particle observables. For instance, only spin or only position degrees of freedom.
1. Identical Particles

2. Statement of the Problem

3. Entanglement for Identical Particles

4. Examples
Two-fermions, $\mathcal{H}^{(1)} = \mathbb{C}^3$

- Basis of $\mathcal{H}^{(1)} = \mathbb{C}^3$: $\{|e_i\rangle, \ i = 1, 2, 3\}$.

- Basis of $\mathcal{H}^{(2)} = \mathbb{C}^3 \wedge \mathbb{C}^3$: $\{|f^i\rangle = \epsilon^{ijk}|e_j \wedge e_k\rangle, \ i = 1, 2, 3\}$.

- 2-particle algebra of observables: $\mathcal{A}^{(2)} \cong M_3(\mathbb{C})$. Indeed, $\mathcal{A}^{(1)} = U(3)$ and $3 \otimes 3 = 6 \oplus \bar{3}$. 

Indistinguishability
Examples

Two-fermions, $\mathcal{H}^{(1)} = \mathbb{C}^3$;

Case 1: $\mathcal{A}_0 = \mathcal{A}^{(1)} \subset \mathcal{A}^{(2)}$

For any $|\psi\rangle \in \mathcal{H}^{(2)}$, the pure state $\omega_\psi = |\psi\rangle \langle \psi|$ restricted to $\mathcal{A}_0 = \mathcal{A}^{(1)} \subset \mathcal{A}^{(2)}$ gives zero entropy.

Indeed, the $U(3)$ representation $\bar{3}$ is irreducible.

However the entropy computed by partial trace is equal to $\log 2$ for any $|\psi\rangle$!  G. Ghirardi and L. Marinatto, Phys. Rev. A, 70, 012109 (2004).
Examples

Two-fermions, $\mathcal{H}^{(1)} = \mathbb{C}^3$;

Case 2: Partial Observations

- $A_0 \subset A^{(2)}$ generated by
  \[ M^{ij} = |f^i\rangle\langle f^j|, \quad i, j = 1, 2, \quad \text{and} \quad 1_{3 \times 3}. \]

- (Pure) states on $A^{(2)}$:
  \[ \omega_\theta = |\psi_\theta\rangle\langle\psi_\theta|, \quad \text{with} \]
  \[ |\psi_\theta\rangle = \cos \theta |f^1\rangle + \sin \theta |f^2\rangle. \]
  \[ (21) \quad (22) \]
Examples

**Two-fermions, \( \mathcal{H}^{(1)} = \mathbb{C}^3 \);**

**Case 2: Partial Observations**

We obtain the following results:

1. **\( 0 < \theta < \frac{\pi}{2} \):** The null vector space is trivial and
   
   \[
   S(\theta) = -\cos^2 \theta \log \cos^2 \theta - \sin^2 \theta \log \sin^2 \theta \tag{23}
   \]

2. **\( \theta = 0 \):** Null vector space is non-trivial, \( \dim \mathcal{H} = 2 \) and entropy is zero.

3. **\( \theta = \frac{\pi}{2} \):** Null vector space is non-trivial, \( \dim \mathcal{H} = 1 \) and entropy is zero.
Examples

Double-Well, $\mathcal{H}^{(1)} = \mathbb{C}^4$

- $\mathcal{H}^{(1)}$ for a fermion with two external d.o.f. (position) and two internal d.o.f. (spin):

  \begin{align*}
  a_\sigma, a_\sigma^\dagger & : \text{“left” with spin } \sigma = +, - \quad (24) \\
  b_\sigma, b_\sigma^\dagger & : \text{“right” with spin } \sigma = +, - \quad (25)
  \end{align*}

- $\mathcal{H}^{(2)}$ spanned by $a_1^\dagger a_2^\dagger |\Omega\rangle$, $b_1^\dagger b_2^\dagger |\Omega\rangle$, $a_\sigma^\dagger b_\rho^\dagger |\Omega\rangle$, where $|\Omega\rangle$ is the vacuum.

- 2-particles algebra of observables $\mathcal{A} = M_6(\mathbb{C})$. 

Indistinguishability
Double-Well, $\mathcal{H}^{(1)} = \mathbb{C}^4$

- Consider the family of (pure) states $\omega_\theta = |\psi_\theta\rangle\langle\psi_\theta|$ on $\mathcal{A}$ with

$$|\psi_\theta\rangle = \left( \cos \theta \ a_1^\dagger b_2^\dagger + \sin \theta \ a_2^\dagger b_1^\dagger \right) |\Omega\rangle \quad (26)$$

- Subalgebra $\mathcal{A}_0$: one-particle observations at the left position. It is generated by

$$\mathbb{1}_\mathcal{A},$$
$$n_{12} = a_1^\dagger a_1 a_2^\dagger a_2,$$
$$N_a = a_1^\dagger a_1 + a_2^\dagger a_2,$$
$$T_i = \frac{1}{2} a_\sigma^\dagger (\sigma_i)^{\sigma\sigma'} a_{\sigma'}$$
Examples

Two Fermions, $\mathcal{H}^{(1)} = \mathbb{C}^4$

We obtain the following results:

1. $0 < \theta < \frac{\pi}{2}$: non-trivial null vector space, $\dim \mathcal{H}_\theta = 4$ and

   $$S(\theta) = - \cos^2 \theta \log \cos^2 \theta - \sin^2 \theta \log \sin^2 \theta.$$

2. $\theta = 0$: non-trivial null vector space, $\dim \mathcal{H}_\theta = 2$, entropy is zero.

3. $\theta = \frac{\pi}{2}$: non-trivial null vector space, $\dim \mathcal{H}_\theta = 2$, entropy is zero.

The last two cases should be contrasted with $S = \log 2$ found in the literature.  
Two Bosons, $\mathcal{H}^{(1)} = \mathbb{C}^3$

- Basis of $\mathcal{H}^{(1)} = \mathbb{C}^3$: $\{|e_i\rangle, \ i = 1, 2, 3\}$.

- Basis of $\mathcal{H}^{(2)} = \mathbb{C}^3 \vee \mathbb{C}^3$:

$$|e_i \lor e_j\rangle = \begin{cases} 
\frac{1}{\sqrt{2}} \left( |e_i\rangle \otimes |e_j\rangle + |e_j\rangle \otimes |e_i\rangle \right), & i \neq j, \\
|e_i\rangle \otimes |e_i\rangle 
\end{cases}$$

2-particle algebra of observables: $\mathcal{A}^{(2)} \cong M_6(\mathbb{C})$. Indeed, $\mathcal{A}^{(1)} = U(3)$ and $3 \otimes 3 = 6 \oplus \bar{3}$.

- A pure state on $\mathcal{A}^{(2)}$:

$$|\theta, \varphi\rangle = \sin \theta \cos \varphi \ |e_1 \lor e_2\rangle + \sin \theta \sin \varphi \ |e_1 \lor e_3\rangle + \cos \theta \ |e_3 \lor e_3\rangle$$
Two Bosons, $\mathcal{H}^{(1)} = \mathbb{C}^3$

- $A_0$ are generated by $|e_i\rangle\langle e_j|$, with $i = 1, 2$ and $1 \sim 6$. This is equivalent to restriction to one-particle observables associated with only $|e_1\rangle$ and $|e_2\rangle$. The observables are equivalent to $U(2)$ (or better $SU(2)$).

- We may split the $\mathcal{H} = \mathbb{C}^6$ into invariant subspaces w.r.t. $SU(2)$ as $\mathbb{C}^6 = \mathbb{C}^3 \oplus \mathbb{C}^2 \oplus \mathbb{C}$, or equivalently $(1) \oplus (1/2) \oplus (0)$.

- For $(\theta, \varphi) \neq (0, 0)$, the GNS construction leads to a cyclic state $|\tilde{1}_A\rangle = |e_1 \lor e_1\rangle + |e_1 \lor e_3\rangle + |e_3 \lor e_3\rangle$, with entropy

$$S(\theta, \varphi) = -2 \cos^2 \theta \log \cos \theta$$

$$- 2 \sin^2 \theta \left[ \cos^2 \varphi \log(\sin \theta \cos \varphi) + \sin \varphi \log(\sin \theta \sin \varphi) \right]$$
As Bal says:

Think quantumly,  
Act Planckly!
MUITO OBRIGADO