Fernando Falceto
Departamento de Física Teórica.
Universidad de Zaragoza

An introduction to quantum circuits.

Martes cuantico, March 22, 2016

Why to study quantum circuits?

- An interesting way of looking at quantum mechanics.
- Miniaturization makes it necessary.
- Interesting new algorithms.
- New possibilities to simulate quantum systems.

Why to study quantum circuits?

- An interesting way of looking at quantum mechanics.
- Miniaturization makes it necessary.
- Interesting new algorithms.
- New possibilities to simulate quantum systems.

Why to study quantum circuits?

- An interesting way of looking at quantum mechanics.
- Miniaturization makes it necessary.
- Interesting new algorithms.
- New possibilities to simulate quantum systems.

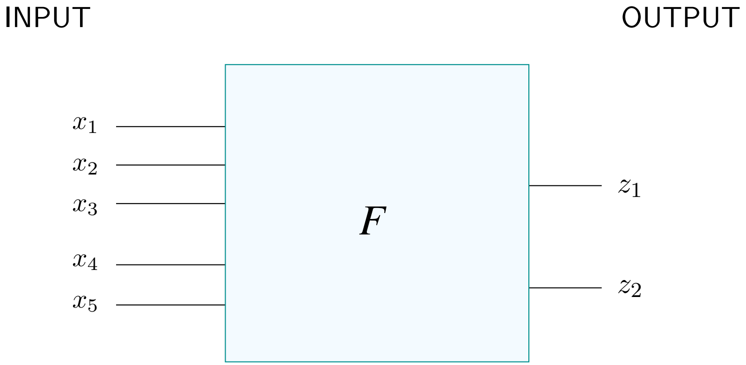
Why to study quantum circuits?

- An interesting way of looking at quantum mechanics.
- Miniaturization makes it necessary.
- Interesting new algorithms.
- New possibilities to simulate quantum systems.

Why to study quantum circuits?

- An interesting way of looking at quantum mechanics.
- Miniaturization makes it necessary.
- Interesting new algorithms.
- New possibilities to simulate quantum systems.

Classical circuits.



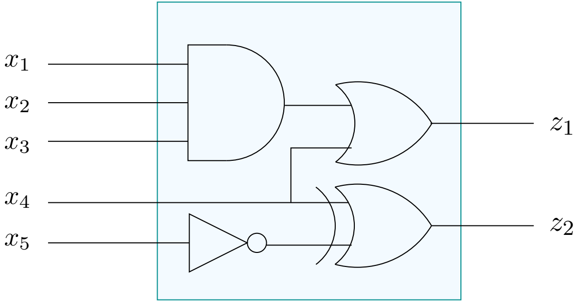
Variables x_i, z_j take values 1 (true) and 0 (false).

This circuit represents a *boolean* function

$$F : \{0,1\}^5 \longrightarrow \{0,1\}^2$$

Classical circuits.

INPUT OUTPUT



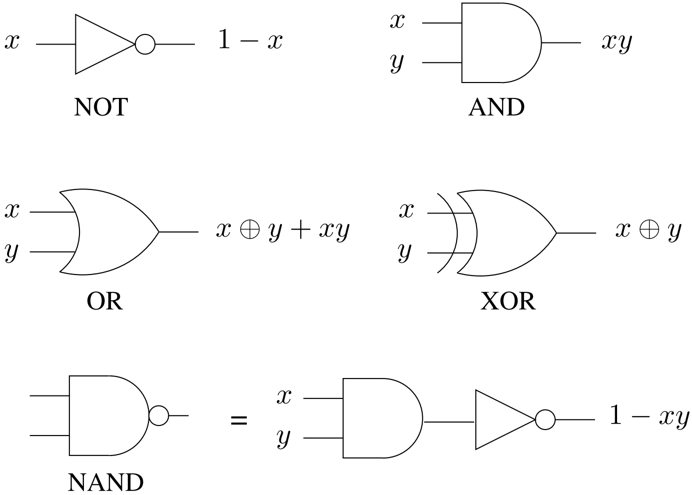
Variables x_i, z_j take values 1 (true) and 0 (false).

This circuit represents a *boolean* function

$$F : \{0,1\}^5 \longrightarrow \{0,1\}^2$$

Classical logic circuits are made of wires and gates.

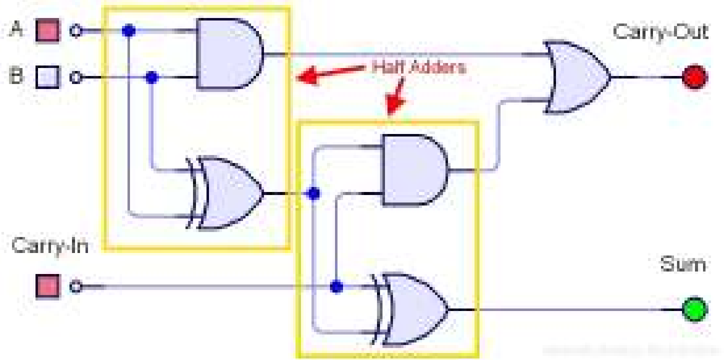
Some classical gates.



$$x \oplus y = x + y \pmod 2.$$

Some classical circuits.

The full adder: adds two bits with carries.

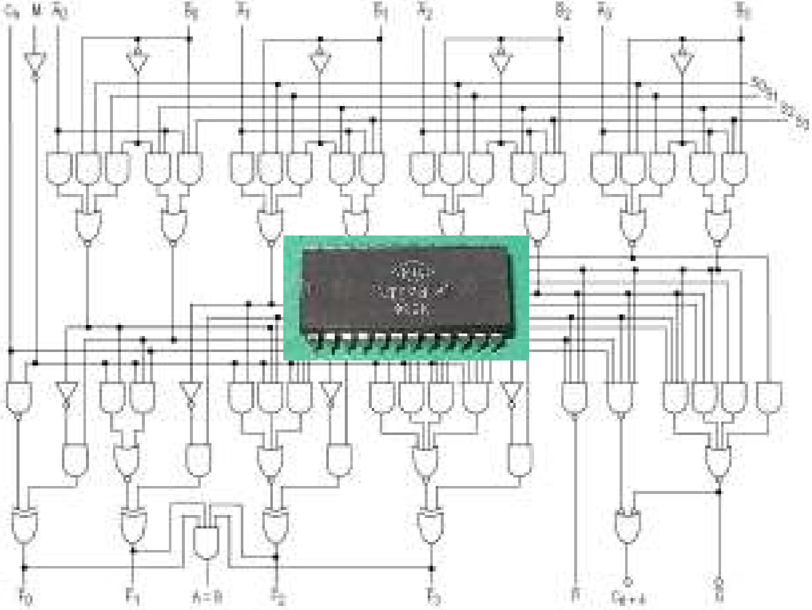


Some classical circuits.

1968: SN74LS181, first Arithmetic Logic Unit in a single chip.

Able to performs 32 different operations with four-bit numbers.

Some classical circuits.

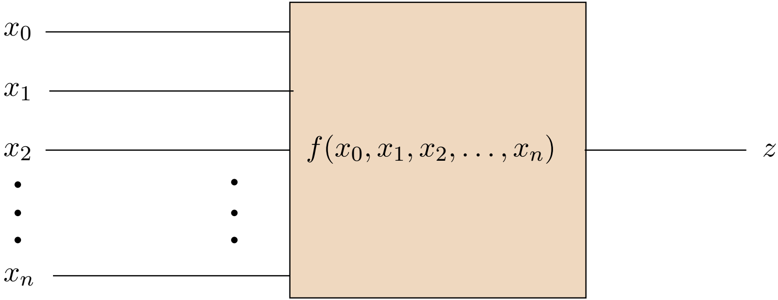


Universal classical gates.

Which logic gates do we need to build any classical circuit?

Consider a general boolean function in $n+1$ variables

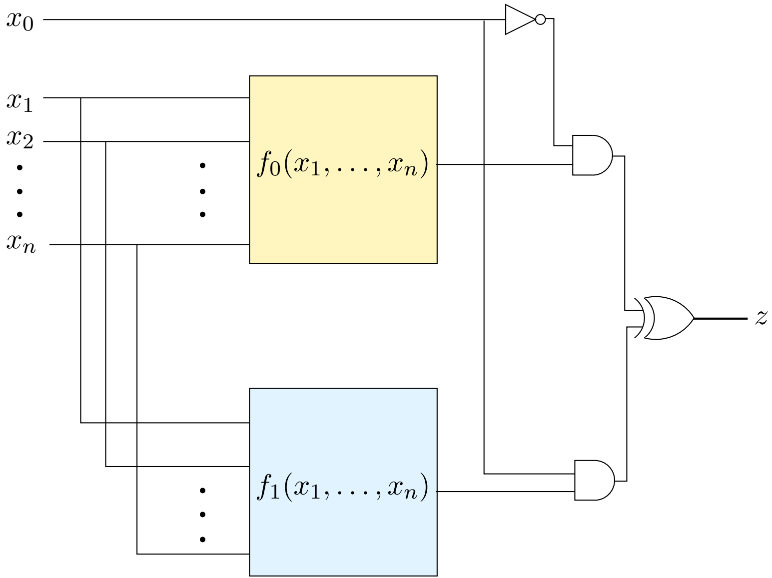
$$f : \{0,1\}^{n+1} \longrightarrow \{0,1\}$$



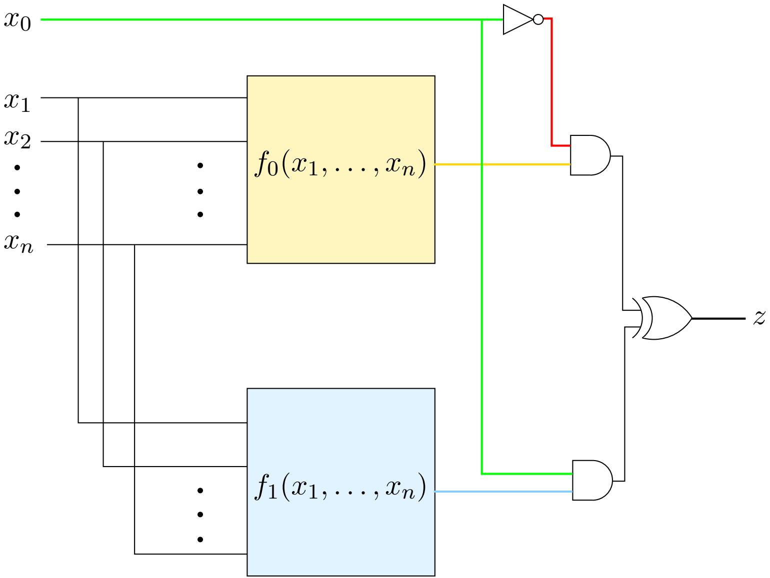
We call: $f_0(x_1, x_2, \ldots, x_n) = f(0, x_1, x_2, \ldots, x_n)$

$f_1(x_1, x_2, \ldots, x_n) = f(1, x_1, x_2, \ldots, x_n)$. Then...

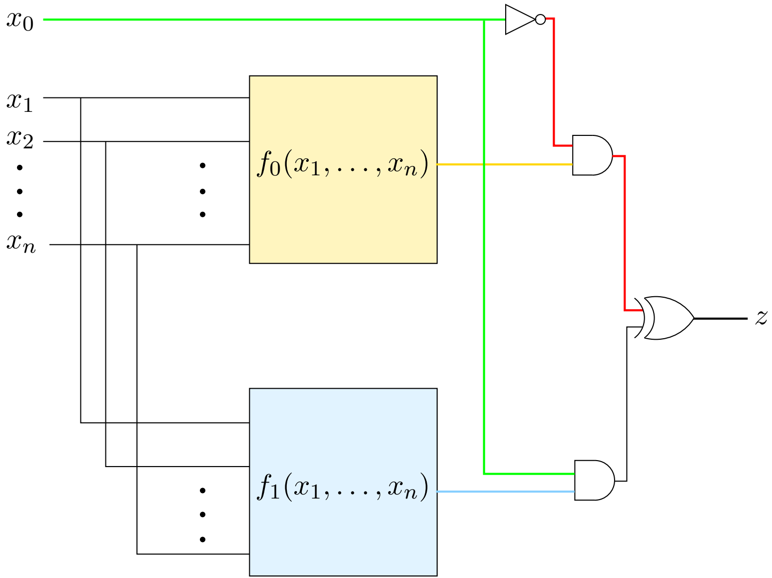
Universal classical gates. $f_{x_0}(x_1, \ldots, x_n) = f(x_0, x_1, \ldots, x_n)$



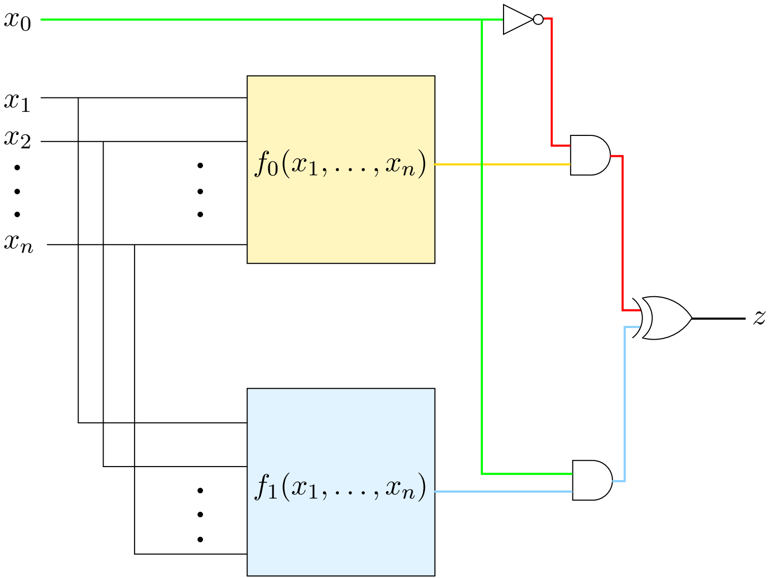
Universal classical gates. $f_{x_0}(x_1, \ldots, x_n) = f(x_0, x_1, \ldots, x_n)$



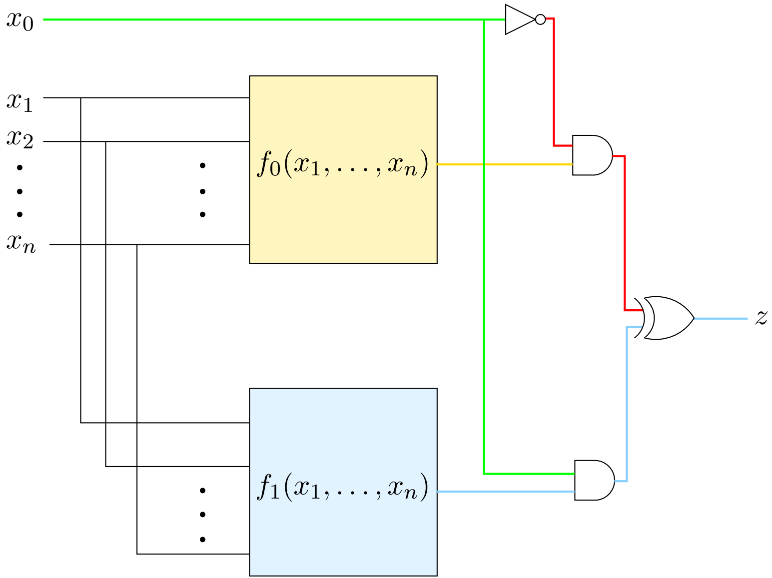
Universal classical gates. $f_{x_0}(x_1, \ldots, x_n) = f(x_0, x_1, \ldots, x_n)$



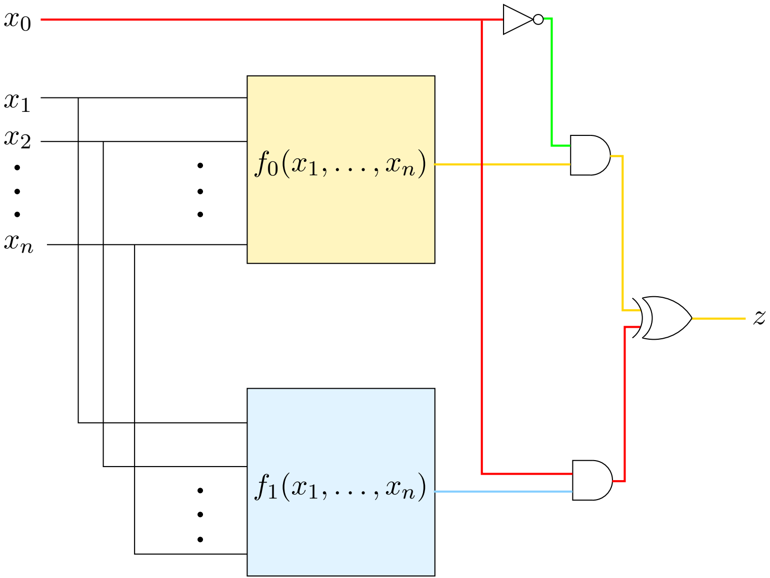
Universal classical gates. $f_{x_0}(x_1, \ldots, x_n) = f(x_0, x_1, \ldots, x_n)$



Universal classical gates. $f_{x_0}(x_1, \ldots, x_n) = f(x_0, x_1, \ldots, x_n)$

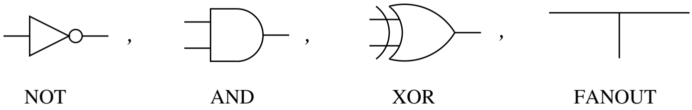


Universal classical gates. $f_{x_0}(x_1, \ldots, x_n) = f(x_0, x_1, \ldots, x_n)$



Universal classical gates.

Then



form a universal set of gates.

Every boolean function

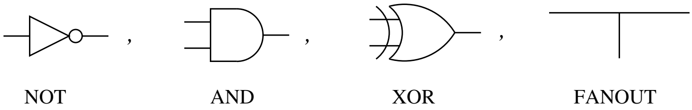
$$f : \underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{n} \longrightarrow \{0,1\}$$

can be iteratively constructed from these gates.

And, therefore, any $F : \{0,1\}^n \longrightarrow \{0,1\}^m$.

Universal classical gates.

Then



form a universal set of gates.

Every boolean function

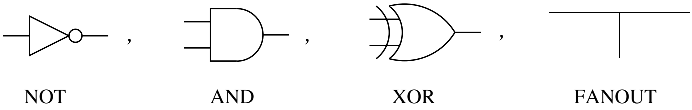
$$f : \underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{n} \longrightarrow \{0,1\}$$

can be iteratively constructed from these gates.

And, therefore, any $F : \{0,1\}^n \longrightarrow \{0,1\}^m$.

Universal classical gates.

Then



form a universal set of gates.

Every boolean function

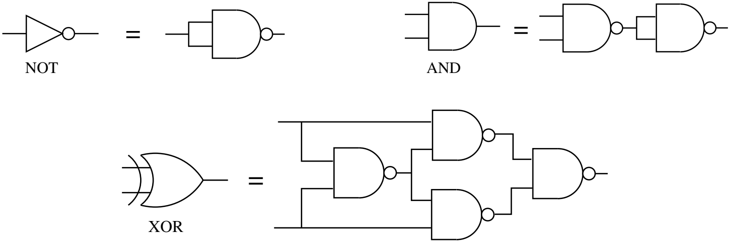
$$f : \underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{n} \longrightarrow \{0,1\}$$

can be iteratively constructed from these gates.

And, therefore, any $F : \{0,1\}^n \longrightarrow \{0,1\}^m$.

Universal classical gates.

But:

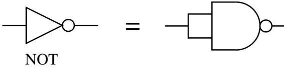


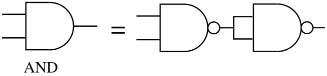
can be obtained combining only

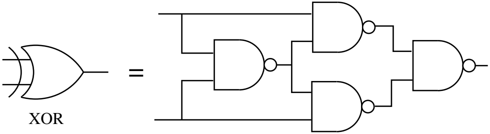
They form a universal set of gates for classical circuits.

Universal classical gates.

But:







can be obtained combining only

Peirce, 1881

Sheffer, 1913

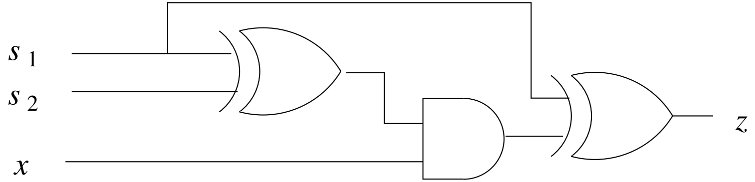
They form a universal set of gates for classical circuits.

Programmable classical circuit

Design a circuit that can compute any $f : \{0,1\}^n \longrightarrow \{0,1\}$.

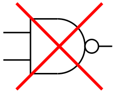
One would need 2^n selection bits.

For $n = 1$



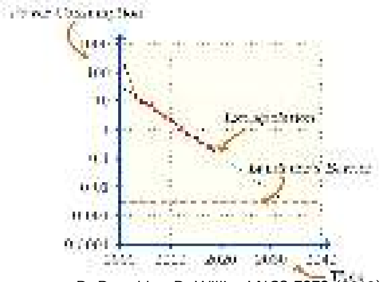
		Selection bits: s_1, s_2			
		0,0	0,1	1,0	1,1
x	0	0	0	1	1
	1	0	1	0	1
		$z = 0$	$z = x$	$z = \overline{x}$	$z = 1$

Reversible logic circuits.

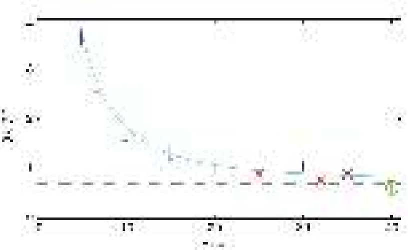
NAND gate is not reversible 

$$F : \{0,1\}^n \longleftrightarrow \{0,1\}^n$$

1. Landauer principle (1961): For every bit of information lose in a circuit a minimum heat of $kT \ln 2$ is disipated.



R. Dreschler, R. Willie. LNCS **7373** (2012)

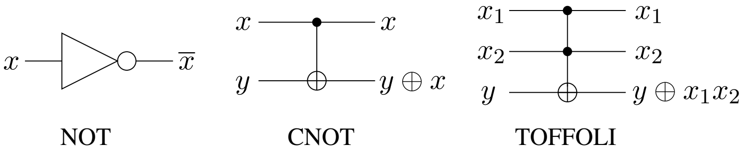


A. Bérut et al. Nature **483** (2012)

2. Quantum circuits are reversible.

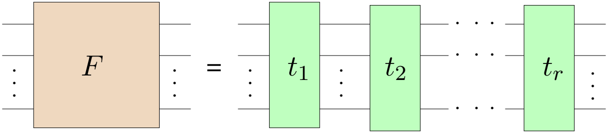
Reversible logic circuits.

Reversible gates:



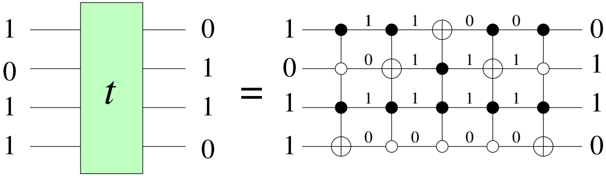
Invertible functions,

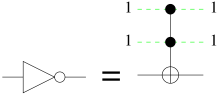
$$F : \{0,1\}^n \longleftrightarrow \{0,1\}^n$$

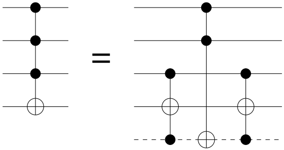


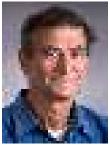
are a product of transpositions: $F = t_r \dots t_2 t_1$.

Reversible logic circuits.

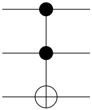







Toffoli, 1980

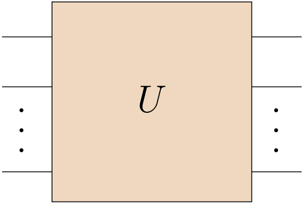
Toffoli gate (+ ancilla bits) universal for reversible classical circuits.



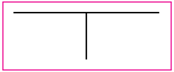
Quantum circuits

Qubit: Smallest (non trivial) quantum system.
Hilbert space $\mathbb{C}^2$ with computational basis $\{|0\rangle, |1\rangle\}$.

Quantum circuits: Unitary operators $U : \mathcal{H} \to \mathcal{H}$ acting on systems composed of several qubits, $\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$.

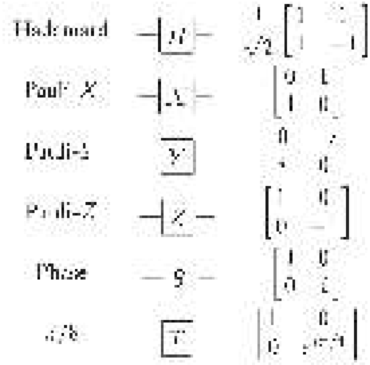


Quantum circuits are reversible.

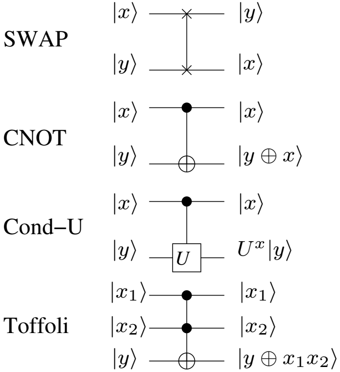
No-cloning theorem: FANOUT gate  not allowed.

Some quantum gates

Single qubit gates

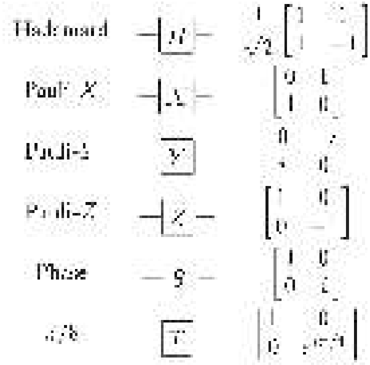


Multiple qubit gates

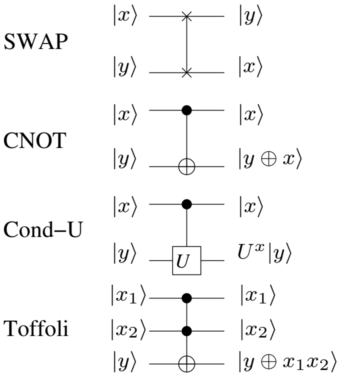


Some quantum gates

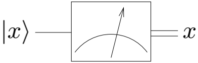
Single qubit gates



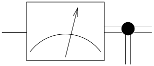
Multiple qubit gates



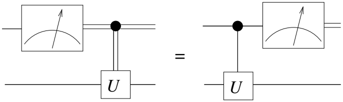
Measurements



Measurement

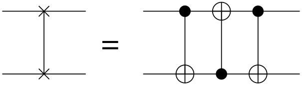


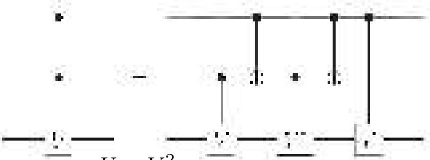
Classical control



Principle of deferred measuremet

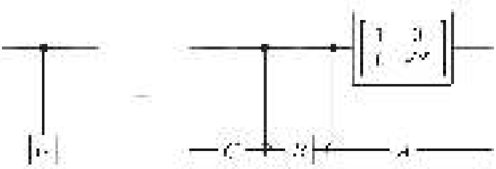
Some useful relations





$U = V^2$

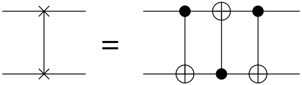
Toffoli from CNOT!!!

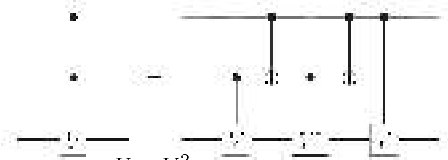


$U = e^{i\alpha} AXBXC$, with

$$ABC = 1, \; X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

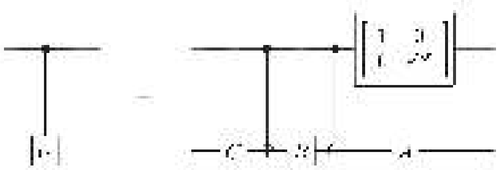
Some useful relations





$U = V^2$

Toffoli from CNOT!!!



$U = e^{i\alpha} AXBXC$, with

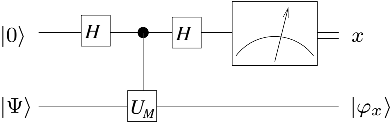
$$ABC = 1,\ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Measuring M

$M = \lambda_0 P_0 + \lambda_1 P_1,$

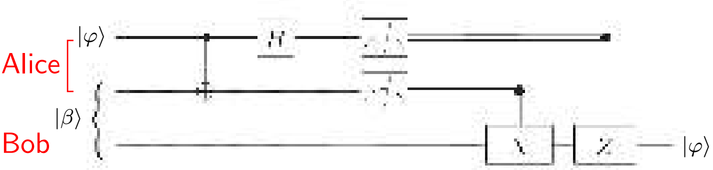
$P_x = |\varphi_x\rangle \langle\varphi_x|\,,$

$U_M = P_0 - P_1$



$\mathrm{pr}(x) = |\langle\varphi_x|\Psi\rangle|^2$

Teleportation



$$|\varphi\rangle = a\,|0\rangle + b\,|1\rangle\,, \qquad |\beta\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Teleportation

$$|\varphi\rangle = a\,|0\rangle + b\,|1\rangle\,, \qquad |\beta\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|00\rangle + |11\rangle)\right]$$

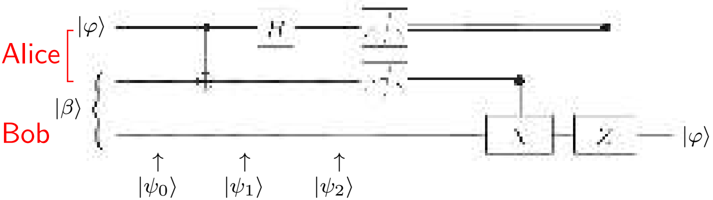
Teleportation

$$|\varphi\rangle = a\,|0\rangle + b\,|1\rangle\,, \qquad |\beta\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|00\rangle + |11\rangle)\right]$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|10\rangle + |01\rangle)\right]$$

Teleportation



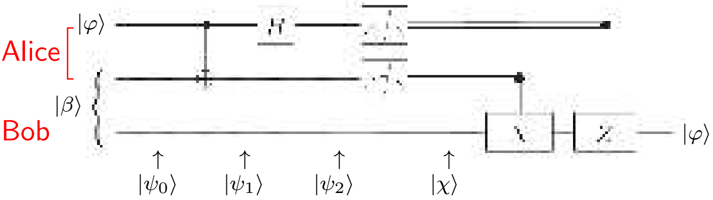
$$|\varphi\rangle = a\,|0\rangle + b\,|1\rangle\,, \qquad |\beta\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|00\rangle + |11\rangle)\right]$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|10\rangle + |01\rangle)\right]$$

$$\begin{aligned}|\psi_2\rangle &= \tfrac{1}{2}\left[a(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + b(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\right] \\ &= \tfrac{1}{2}\left[|00\rangle\,(a\,|0\rangle + b\,|1\rangle) + |01\rangle\,(a\,|1\rangle + b\,|0\rangle)\right. \\ &\qquad \left. + |10\rangle\,(a\,|0\rangle - b\,|1\rangle) + |11\rangle\,(a\,|1\rangle - b\,|0\rangle)\right]\end{aligned}$$

Teleportation



$$|\varphi\rangle = a\,|0\rangle + b\,|1\rangle\,, \qquad |\beta\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|00\rangle + |11\rangle)\right]$$

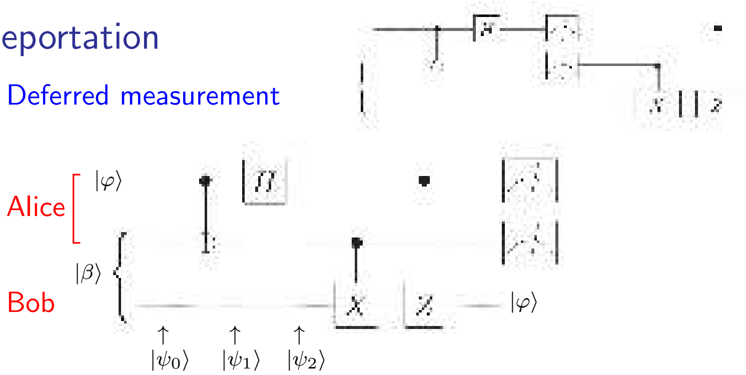
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|10\rangle + |01\rangle)\right]$$

$$\begin{aligned}|\psi_2\rangle &= \tfrac{1}{2}\left[a(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + b(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\right] \\ &= \tfrac{1}{2}\left[|00\rangle\,(a\,|0\rangle + b\,|1\rangle) + |01\rangle\,(a\,|1\rangle + b\,|0\rangle)\right. \\ &\quad \left. + |10\rangle\,(a\,|0\rangle - b\,|1\rangle) + |11\rangle\,(a\,|1\rangle - b\,|0\rangle)\right]\end{aligned}$$

$$\begin{aligned}&00 \mapsto |\chi\rangle = a\,|0\rangle + b\,|1\rangle\,, \quad 01 \mapsto |\chi\rangle = a\,|1\rangle + b\,|0\rangle \\ &10 \mapsto |\chi\rangle = a\,|0\rangle - b\,|1\rangle\,, \quad 11 \mapsto |\chi\rangle = a\,|1\rangle - b\,|0\rangle\end{aligned}$$

Teleportation

Deferred measurement

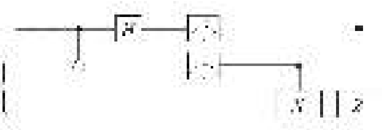


$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|00\rangle + |11\rangle)\right]$$

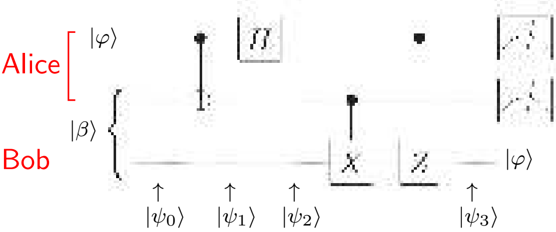
$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|10\rangle + |01\rangle)\right]$$

$$\begin{aligned} |\psi_2\rangle &= \tfrac{1}{2} \left[a(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + b(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\right] \\ &= \tfrac{1}{2} \left[|00\rangle\,(a\,|0\rangle + b\,|1\rangle) + |01\rangle\,(a\,|1\rangle + b\,|0\rangle) \right. \\ &\quad \left. + |10\rangle\,(a\,|0\rangle - b\,|1\rangle) + |11\rangle\,(a\,|1\rangle - b\,|0\rangle)\right] \end{aligned}$$

Teleportation



Deferred measurement



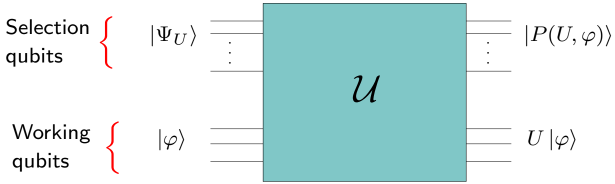
$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|00\rangle + |11\rangle)\right]$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \left[a\,|0\rangle\,(|00\rangle + |11\rangle) + b\,|1\rangle\,(|10\rangle + |01\rangle)\right]$$

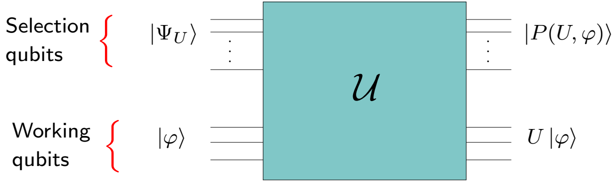
$$\begin{aligned} |\psi_2\rangle &= \frac{1}{2} \left[a(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + b(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\right] \\ &= \frac{1}{2} \left[|00\rangle\,(a\,|0\rangle + b\,|1\rangle) + |01\rangle\,(a\,|1\rangle + b\,|0\rangle) \right. \\ &\quad \left. + |10\rangle\,(a\,|0\rangle - b\,|1\rangle) + |11\rangle\,(a\,|1\rangle - b\,|0\rangle)\right] \end{aligned}$$

$$|\psi_3\rangle = \frac{1}{2}\,(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\,(a\,|0\rangle + b\,|1\rangle)$$

Programmable quantum computer?



Programmable quantum computer?

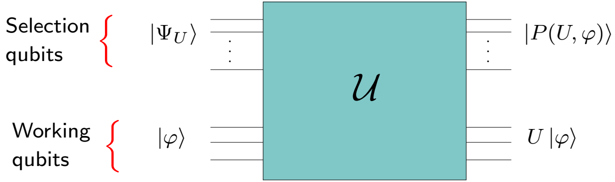


1. P does not depend on φ.

$$\langle\varphi|\varphi'\rangle = \langle\Psi_U|\Psi_U\rangle\langle\varphi|\varphi'\rangle = \langle P(U,\varphi)|P(U,\varphi')\rangle\langle\varphi|U^\dagger U|\varphi'\rangle$$

Then $\langle P(U,\varphi)|P(U,\varphi')\rangle = 1$ which implies $|P(U,\varphi)\rangle = |P(U,\varphi')\rangle$

Programmable quantum computer?



1. P does not depend on φ.

$$\langle\varphi|\varphi'\rangle = \langle\Psi_U|\Psi_U\rangle\langle\varphi|\varphi'\rangle = \langle P(U,\varphi)|P(U,\varphi')\rangle\langle\varphi|U^\dagger U|\varphi'\rangle$$

Then $\langle P(U,\varphi)|P(U,\varphi')\rangle = 1$ which implies $|P(U,\varphi)\rangle = |P(U,\varphi')\rangle$

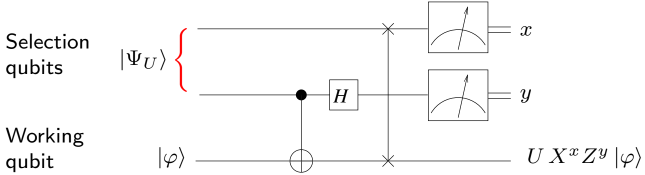
2. If $U' \neq \mathrm{e}^{i\alpha}U$ then $\langle\Psi_U|\Psi_{U'}\rangle = 0$

$$\langle\Psi_U|\Psi_{U'}\rangle = \langle\Psi_U|\Psi_{U'}\rangle\langle\varphi|\varphi\rangle = \langle P(U)|P(U')\rangle\langle\varphi|U^\dagger U'|\varphi\rangle$$

Then, either $U^\dagger U' = \mathrm{e}^{i\alpha}I$ or $\langle P(U)|P(U')\rangle = \langle\Psi_U|\Psi_{U'}\rangle = 0$

Already, for one working qubit we have infinitely many *different* unitary operators, i.e. we need inifinitely many selection qubits.

Stochastic programmable quantum computer



$$\Psi_U = \frac{1}{\sqrt{2}}(U \otimes I)(|00\rangle + |11\rangle)$$

With a probability $1/4$, when $x = 0$ and $y = 0$, the output is $U\,|\psi\rangle$.

Reading the registers we know when we get the desired result.

For m working qubits $\rightarrow 2m$ selection qubits. Linear dependence!!.

But the probability of success is 2^{-m}.

Universal quantum gates

Any unitary operator can be written as the product of others which are the identity on all but two vectors of the basis

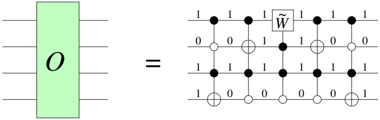
$$U = O_r \cdots O_2 O_1$$

Universal quantum gates

Any unitary operator can be written as the product of others which are the identity on all but two vectors of the basis

$$U = O_r \cdots O_2 O_1$$

Take one such $O = W \oplus I$, s.t. W acts on $<|1011\rangle, |0110\rangle>$



$\tilde{W}$ acts on $<|1110\rangle, |0110\rangle>$

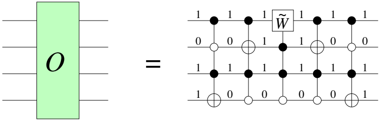
i.e. it is a single qubit gate.

Universal quantum gates

Any unitary operator can be written as the product of others which are the identity on all but two vectors of the basis

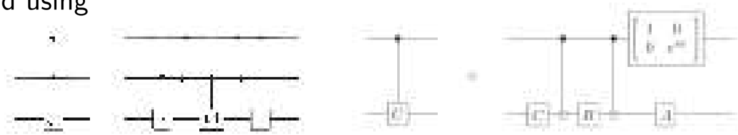
$$U = O_r \cdots O_2 O_1$$

Take one such $O = W \oplus I$, s.t. W acts on $< |1011\rangle\,, |0110\rangle >$



$\tilde{W}$ acts on $< |1110\rangle\,, |0110\rangle >$
i.e. it is a single qubit gate.

And using

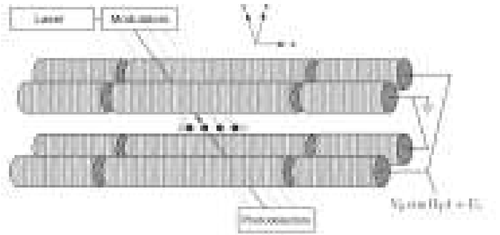


we get that CNOT + one qubit gates form a universal set.

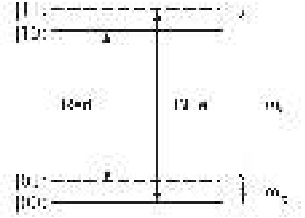
DiVicenzo; Sleator & Weinfurter; Barenco et al. (1995)

Physical realization. Cirac & Zoller (1995)

Trapped ions.

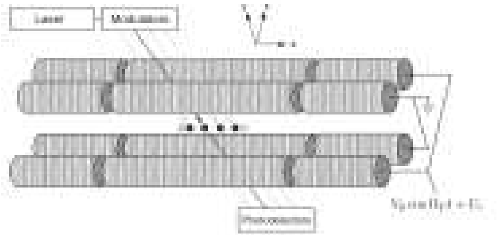


Two energy levels in every ion coupled to a colective phonon.

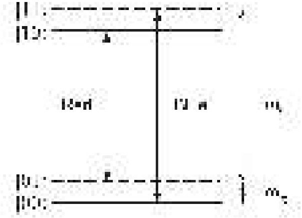


Physical realization. Cirac & Zoller (1995)

Trapped ions.



Two energy levels in every ion coupled to a colective phonon.



Single qubit operators

Apply an electromagnetic field of frequency ω_0 and phase φ. The interaction Hamiltonian is

$$H_I = \frac{\hbar\Omega}{2}(X\cos\varphi + Y\sin\varphi)\otimes I$$

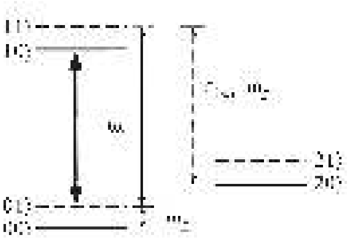
By controlling the phase and time any $U \otimes I$ can be simulated.

Physical realization. Cirac & Zoller (1995)

SWAP (ion-phonon) gate.

With an electromagnetic field of frequency $\omega_0 - \omega_Z$:

$$|01\rangle \leftrightarrow |10\rangle$$

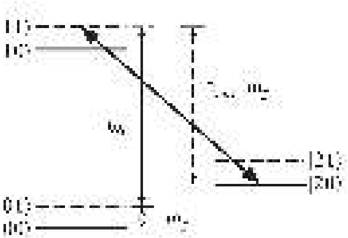


Physical realization. Cirac & Zoller (1995)

SWAP (ion-phonon) gate.

With an electromagnetic field of frequency $\omega_0 - \omega_Z$:

$$|01\rangle \leftrightarrow |10\rangle$$



c-Z (ion-phonon) gate.

With a frequency $\omega_{aux} + \omega_Z$:

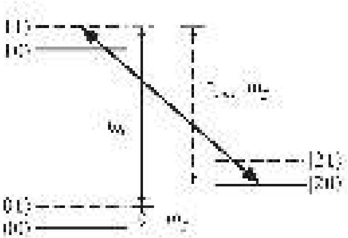
$$|11\rangle \leftrightarrow -|11\rangle$$

Physical realization. Cirac & Zoller (1995)

SWAP (ion-phonon) gate.

With an electromagnetic field of frequency $\omega_0 - \omega_Z$:

$$|01\rangle \leftrightarrow |10\rangle$$

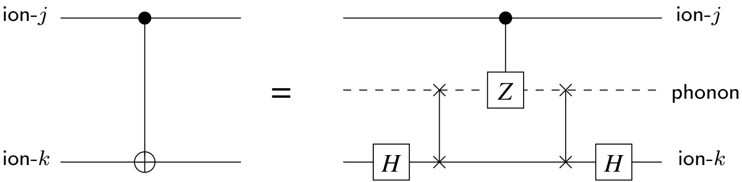


c-Z (ion-phonon) gate.

With a frequency $\omega_{aux} + \omega_Z$:

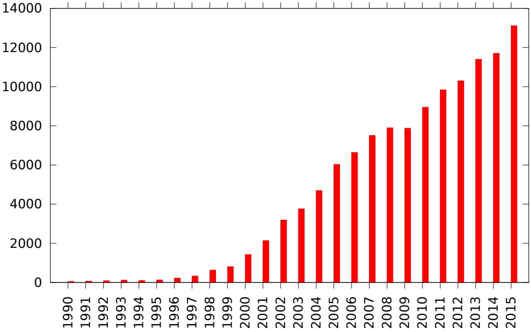
$$|11\rangle \leftrightarrow -|11\rangle$$

CNOT (ion-ion) gate.



The rush starts.

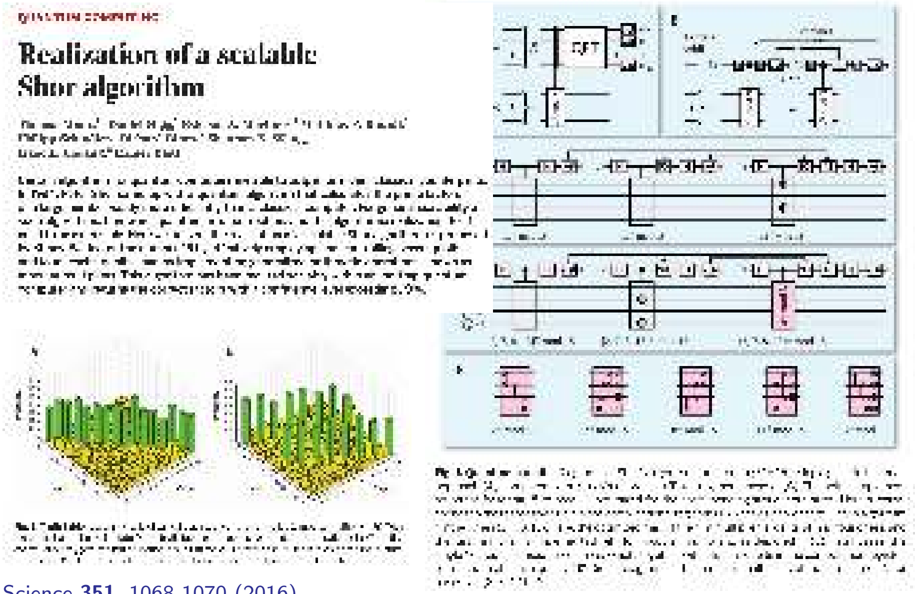
Papers per year related to "Quantum Information".



Source, Google Scholar.

Since 1995, the number of papers on Quantum Information has been growing consistently fast.

Quantum computing



Science **351**, 1068-1070 (2016).

Simulation of quantum systems

Bulk locality and quantum error correction in AdS/CFT

JHEP 04 (2015) 163

Quantum Algorithms for Quantum Field Theories

Science **336**, 1130-1133 (2012)

Digital quantum simulation of fermionic models with a superconducting circuit

Nature Communications **6**, 7654 (2015)

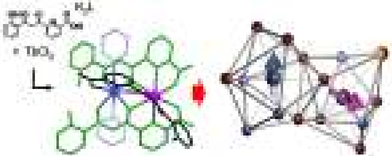
Physical realizations. Science **339** (2013). Special issue

Scaling the Ion Trap Quantum Processor

Topological Quantum Computation—From Basic Concepts to First Experiments

Superconducting Circuits for Quantum Information: An Outlook

Quantum Spintronics: Engineering and Manipulating Atom-Like Spins in Semiconductors



G. Aromí, D. Aguilà, P. Gámez, F. Luis, O. Roubeau, Chem. Soc. Rev. **41**, (2012).

MUCHAS
GRACIAS

MUCHAS
GRACIAS

FELICES VACACIONES